Problem sheet 5

Problem 1 (Nonconvexity of stored energy functions)

Assume that a stored energy function \( W(x, \cdot) \) is convex in its second argument \( F \in \mathbb{R}^{3,3}, \det F > 0 \) for a fixed \( x \in D \). Show that in this case \( W(x, F) \) does not go to infinity as \( \det F \to 0 \).

**Hint:** The set of all matrices \( F \in \mathbb{R}^{3,3}, \det F > 0 \) is not convex. Let \( W^* \) be a convex extension of \( W \) defined on the convex hull \( \text{co}\{F \in \mathbb{R}^{3,3}, \det F > 0\} = \mathbb{R}^{3,3} \). For \( \lambda \in [0,1] \), \( G, H \in \{F \in \mathbb{R}^{3,3}, \det F > 0\} \) consider the function

\[
\omega(\lambda) := W^*(x, (1-\lambda)G + \lambda H).
\]

**Solution**

To see that \( \{F \in \mathbb{R}^{3,3}, \det F > 0\} \) is not convex, consider e.g.

\[
-1 = \frac{1}{2} \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}.
\]

To see that \( \text{co}\{F \in \mathbb{R}^{3,3}, \det F > 0\} = \mathbb{R}^{3,3} \) consider

\[
F = \frac{1}{2}(\lambda I + 2F) + \frac{1}{2}(-\lambda I)
\]

for an arbitrary matrix \( F \in \mathbb{R}^{3,3} \). The second matrix belongs to \( \text{co}\{F \in \mathbb{R}^{3,3}, \det F > 0\} \) for all \( \lambda \) (see convex combination above). For the first matrix \( \det(\lambda I + 2F) \) is a polynomial of degree 3 in \( \lambda \) whose monomial of highest degree is \( \lambda^3 \). Therefore \( \exists \lambda > 0 \) such that \( \det(\lambda I + 2F) > 0 \). Hence every \( F \in \mathbb{R}^{3,3} \) can be written as a convex combination of two matrices in \( \text{co}\{F \in \mathbb{R}^{3,3}, \det F > 0\} \).

Since \( \{F \in \mathbb{R}^{3,3}, \det F > 0\} \) is not convex, \( \exists \mu \in [0,1] \) and matrices \( G, H \in \{F \in \mathbb{R}^{3,3}, \det F > 0\} \) such that \( (1-\mu)G + \mu H \notin \{F \in \mathbb{R}^{3,3}, \det F > 0\} \). Moreover \( \exists \lambda_0 \in [0, \mu] \) such that

\[
\det(1-\lambda)G + \lambda H > 0 \quad 0 \leq \lambda < \lambda_0,
\]

\[
\det(1-\lambda_0)G + \lambda_0 H = 0.
\]
If $W(x,F)$ would go to infinity for $\det F \to 0$, $\omega(\lambda)$ would go to infinity for $\lambda \to \lambda_0$ as well. But due to convexity of $\omega(\lambda)$
\[ \sup_{0 \leq \lambda \leq 1} \omega(\lambda) \leq \max\{\omega(0),\omega(1)\} < \infty. \]

**Problem 2 (Preservation of orientation and injectivity)**

(i) Give an example for a continuous deformation $\phi$ that is orientation-preserving ($\det D\phi > 0$) and has a finite stored hyperelastic energy but is not injective.

(ii) Assume that $\phi = 1 + u$ is differentiable at $x \in D$. Prove that $|Du(x)| < 1$ implies $\det D\phi(x) > 0$.

(iii) Assume $\phi = 1 + u \in C^1(\bar{D}, \mathbb{R}^d)$ and $\bar{D}$ convex. Prove that $\sup_{x \in \bar{D}} |Du(x)| < 1$ implies injectivity of $\phi$.

**Solution**

(i) Consider the following domain, described in polar coordinate system. Here $r \in [1, 2]$, $\theta \in [0, \frac{3\pi}{2}]$. Define a deformation by switching to the polar coordinate system and then scaling the angle:
\[ \phi(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} r \\ \theta \end{pmatrix} \]
This deformation fulfills the requirements but creates an overlap in the regime $[1, 2] \times [2\pi, 3\pi]$.

(ii) Using the Neumann series and because of $|Du(x)| < 1$ all matrices $(1 + tDu(x))$, $0 \leq t \leq 1$ are invertible; thus $\det(1 + tDu(x)) \neq 0$. On the other hand
\[ \delta(t) := \det(1 + tDu(x)) \]
is a continuous function and therefore has a closed image $\delta([0,1]) \subset \mathbb{R}$ which includes $1 = \delta(0)$ but not $0$. It follows that $\delta(1)$ must be positive.

(iii) Let $x_1, x_2 \in \bar{D}$. The mean value theorem applied to all convex combinations of $x_1$ and $x_2$, i.e. all $x$ on the line $\|x_1, x_2\| := \{x | x = tx_1 + (1-t)x_2, t \in (0,1)\}$ yields
\[ |(\phi(x_1) - \phi(x_2)) - (x_1 - x_2)| = |u(x_1) - u(x_2)| \leq \sup_{x \in \|x_1, x_2\|} |Du(x)||x_1 - x_2| \]
\[ < |x_1 - x_2| \text{ if } x_1 \neq x_2. \]
Therefore $x_1 \neq x_2 \Rightarrow \phi(x_1) \neq \phi(x_2)$.

**Problem 3 (Finite Element discretization)**

Give a FE formulation of Newton’s method minimizing
\[ E[\varphi] = \int_D |D\varphi|^2 + \Gamma(\det D\varphi) - f \cdot \varphi \quad \text{s.t. } \varphi|_{\Gamma_D} = 1, \quad D\varphi \cdot n|_{\Gamma_N} = 0, \]
where $\partial D = \Gamma_D \cup \Gamma_N$. 

Solution

Recall

\[
\begin{align*}
\partial_F \det F(A) &= \det F \text{tr}(F^{-1}A) = \text{Cof } F : A \\
\partial_F F^{-1}(A) &= -F^{-1} A F^{-1} \\
\sim \partial_F \text{Cof } F(A) &= \frac{1}{\text{det } F} \left( (\text{Cof } F : A) \text{Cof } F - (\text{Cof } F) A^T (\text{Cof } F) \right)
\end{align*}
\]

With this,

\[
E'[\varphi](\vartheta) = \int D\varphi : D\vartheta + \Gamma' (\det D\varphi) \text{Cof } D\varphi : D\vartheta - f \cdot \vartheta
\]

\[
E''[\varphi](\vartheta, \eta) = \int D\vartheta : D\vartheta
\]

\[
+ \Gamma'' (\det D\varphi) \left( (\text{Cof } D\varphi : D\vartheta) (\text{Cof } D\varphi : D\eta) \\
+ \Gamma' (\det D\varphi) / \det D\varphi \left( (\text{Cof } D\varphi : D\eta) (\text{Cof } D\varphi : D\vartheta) \\
- \Gamma' (\det D\varphi) / \det D\varphi \left( (\text{Cof } D\varphi) D\eta^T : (D\vartheta \text{Cof } D\varphi^T)
\right)
\right)
\]

Observe that the integrands vanish outside of \(\text{supp } \vartheta\) or, as the case may be, \(\text{supp } \vartheta \cap \text{supp } \eta\).

**Step 1:** For the moment, consider only vanishing Dirichlet boundary values.

Let \((\vartheta_i)\) be basis functions for a finite-dimensional subspace \(V\) of \(H^{1,2}_{\Gamma_D}(D, \mathbb{R})\), e.g. nodal hat functions. Then \((e^k \vartheta_i)\) for Euclidian unity vectors \(e^k, k = 1, 2\), form a basis of the corresponding subspace \(V^2\) of \(H^{1,2}_{\Gamma_D}(D, \mathbb{R}^2)\). Let \(\varphi^k_i\) be the coefficients of \(\varphi \in V^2\) within this basis:

\[
\varphi = e^1 \sum \varphi^1_i \vartheta_i + e^2 \sum \varphi^2_i \vartheta_i
\]

Stacking these \(\varphi^k_i\) into one vector, each entry corresponds to the entry \(E'[\varphi](e^k \vartheta_i)\) in \(\nabla E\):

\[
\varphi = \begin{pmatrix}
\vdots \\
\varphi^1_i \\
\vdots \\
\varphi^2_i \\
\vdots \\
\end{pmatrix} \quad \nabla E[\varphi] = \begin{pmatrix}
\vdots \\
E'[\varphi](e^1 \vartheta_i) \\
\vdots \\
E'[\varphi](e^2 \vartheta_i) \\
\vdots \\
\end{pmatrix}
\]

\[
H E[\varphi] = \begin{pmatrix}
\cdots & E''[\varphi](e^1 \vartheta_i, e^1 \vartheta_j) & \cdots & E''[\varphi](e^1 \vartheta_i, e^2 \vartheta_j) & \cdots \\
\cdots & \vdots & \cdots & \vdots & \cdots \\
\cdots & E''[\varphi](e^2 \vartheta_i, e^1 \vartheta_j) & \cdots & E''[\varphi](e^2 \vartheta_i, e^2 \vartheta_j) & \cdots \\
\cdots & \vdots & \cdots & \vdots & \cdots \\
\end{pmatrix}
\]

An assembly of the matrix’ or vector’s entries would iterate over all elements that belong to \(\text{supp } \vartheta\) or \(\text{supp } \vartheta \cap \text{supp } \eta\) and apply a quadrature rule there. The first summand can be exactly integrated by midpoint quadrature, the second would be approximated up to \(O(h^2)\) by this rule.
Now each step of Newton’s method reads

$$\phi^{(\ell+1)} = \phi^{(\ell)} + p^{(\ell)}, \quad \text{where} \quad HE[\phi^{(\ell)}] p^{(\ell)} = -\nabla E[\phi^{(\ell)}]$$

**Step 2:** Any $\tilde{\phi}$ with $\tilde{\phi}|_{\Gamma_D} = 1$ can be composed as $\tilde{\phi} = \phi + b$, where $\phi$ has homogenous Dirichlet boundary values and $b$ is zero everywhere except on $\Gamma_D$, where it is $1$.

Formally spoken, we need an additional vector space $V_{bd}$ spanned by all nodal basis functions for nodes discretizing $\Gamma_D$. Then $E$, $\nabla E$ and $HE$ are defined on $V^2 \oplus V_{bd}^2$. This modifies our $\ell$ loop as:

$$\phi^{(\ell+1)} = \phi^{(\ell)} + p^{(\ell)}, \quad \text{where} \quad HE[\phi^{(\ell)} + b] p^{(\ell)} = -\nabla E[\phi^{(\ell)} + b]$$

More algorithmically, one may

(i) extend the vector $\phi = (\phi^k_j)$ by values for the Dirichlet boundary nodes and erase the corresponding entries in $p$ before updating $\phi^{(\ell)}$ or

(ii) introduce an auxiliary vector $\tilde{\phi}$ representing an element in $V^2 \oplus V_{bd}^2$. The summation $\phi + b$ is then done by filling one part of $\tilde{\phi}$’s entries from $\phi$ and the other part from $b$. 