Tutorial Numerical Simulation (V4E2)
Summer term 2010
Prof. Dr. M. Rumpf – St. W. von Deylen, B. Geihe

Problem sheet 5
17.05.2010

Problem 1 (Nonconvexity of stored energy functions)
Assume that a stored energy function \( W(x, \cdot) \) is convex in its second argument \( F \in \mathbb{R}^{3,3}, \det F > 0 \) for a fixed \( x \in D \). Show that in this case \( W(x, F) \) does not go to infinity as \( \det F \to 0 \).

**Hint:** The set of all matrices \( F \in \mathbb{R}^{3,3}, \det F > 0 \) is not convex. Let \( W^* \) be a convex extension of \( W \) defined on the convex hull \( \text{co}\{F \in \mathbb{R}^{3,3}, \det F > 0\} = \mathbb{R}^{3,3} \). For \( \lambda \in [0,1], G, H \in \{F \in \mathbb{R}^{3,3}, \det F > 0\} \) consider the function
\[
\omega(\lambda) := W^*(x, (1-\lambda)G + \lambda H).
\]

Problem 2 (Preservation of orientation and injectivity)

(i) Give an example for a continuous deformation \( \phi \) that is orientation-preserving \( (\det D\phi > 0) \) and has a finite stored hyperelastic energy but is not injective.

(ii) Assume that \( \phi = 1 + u \) is differentiable at \( x \in D \). Prove that \( |Du(x)| < 1 \) implies \( \det D\phi(x) > 0 \).

(iii) Assume \( \phi = 1 + u \in C^1(\bar{D}, \mathbb{R}^d) \) and \( \bar{D} \) convex. Prove that \( \sup_{x \in D} |Du(x)| < 1 \) implies injectivity of \( \phi \).

Problem 3 (Finite Element discretization)
Give a FE formulation of Newton’s method minimizing
\[
E[\varphi] = \int_D |D\varphi|^2 + \Gamma(\det D\varphi) - f \cdot \varphi \quad \text{s.t.} \quad \varphi|_{\Gamma_D} = 1, \quad D\varphi \cdot n|_{\Gamma_N} = 0,
\]
where \( \partial D = \Gamma_D \cup \Gamma_N \).