Problem 1 (Existence proof for Lamé-Navier type energies via Lax-Milgram)

Consider the following PDE:

\[ -\text{div} \sigma = f \]

Assume \( f \in (H^{1,2})' \), homogeneous Dirichlet boundary values, no Neumann boundary \((\Gamma_{F,N})\) and an isotropic material, i.e.

\[ \sigma = \lambda \text{tr} \varepsilon[u] \mathbb{1} + 2\mu \varepsilon[u]. \]

(i) Show that the PDE can be rewritten as

\[ -(\lambda + \mu) \nabla \text{div} u - \mu \Delta u = f. \]

(\( \Delta u \) should be understood as by components.)

(ii) Set up a weak formulation for the PDE above.

(iii) Prove existence of a solution directly using the Lax-Milgram theorem. (Do not use Korn’s inequality.)

Solution

(i)

\[
-\text{div} \sigma = - \left( \sum_i \sigma_{ij} \right)_i = - \left( \lambda \sum_i \left( \delta_{ij} \sum_k u_{k,i} \right)_{ij} + \mu \sum_j \left( u_{ij} + u_{ji} \right)_i \right) = -(\lambda + \mu) \nabla \text{div} u - \mu \Delta u
\]

(ii)

\[
l(\vartheta) := \int_D f \cdot \vartheta dx = \int_D \left( -(\lambda + \mu) \nabla \text{div} u - \mu \Delta u \right) \cdot \vartheta dx
\]

\[
= \int_D (\lambda + \mu) \text{div} u \text{div} \vartheta + \mu Du : D\vartheta dx =: a(u, \vartheta)
\]

Weak formulation: Find \( u \in H^{1,2}_{\Gamma_D} \), s.t. \( a(u, \vartheta) = l(\vartheta) \) \( \forall \vartheta \in H^{1,2}_{\Gamma_D} \).
Problem 2 (Existence proof for linearized elasticity via direct method)

The Lax-Milgram theorem is, where applicable, of course a more convenient method than the direct method in the calculus of variations. Nevertheless, instead of showing the existence of weak solution for the Euler-Lagrange equations, you may also tackle these equations as minimum problems via the direct method:

Consider the elastic energy

\[ E_{\text{lin}}[u] = \int_D (C \varepsilon[u] : \varepsilon[u] - f u) \, dx, \quad u \in H^{1,2}_{1,D}(D) \]

with natural Neumann boundary condition, and a minimizing sequence \( (u^k) \).

(i) Show that \( E_{\text{lin}} \) is convex and lower semi-continuous is \( u \) (in fact, \( E_{\text{lin}} \) is continous, but you will need only lower semi-continuity).

(ii) Using \( A^{(\text{double})} \), show that \( (u^k) \) is bounded in \( H^{1,2}_{1,D}(D) \) and thus contains a weakly converging subsequence.

(iii) With Mazur’s lemma, argue that there are coefficients \( \lambda^k_i \geq 0, \sum \lambda^k_i = 1 \), such that the linear combinations sequence \( \sum \lambda^k_i u^i \) converges to \( \lim u^k \) as \( j \to \infty \).

(iv) Of course, wlog you can assume \( E[u^k] \leq \lim E[u^k] + \varepsilon \) for any fixed \( \varepsilon > 0 \). Use the sequence from step (iii), Fatou’s lemma and \( E_{\text{lin}}’ \)’s lower semicontinuity to show \( E_{\text{lin}}[u] = \lim E_{\text{lin}}[u^k] \).

Solution

ad (i): Continuity: To take a shortcut, \( E_{\text{lin}} \) is norm on \( H^{1,2}_{1,D}(D) \) due to the corollary from Korn’s inequality, and every norm is continous. Afoot: \( u \mapsto Du \) is continous, and integrating over a continous integrand is a continous operation, too.

Convexity: \( u \mapsto \varepsilon[u] \) is a linear map \( H^{1,2} \to L^2 \), thus convex. The integrand \( C \varepsilon : \varepsilon \) is pointwise quadratic (thus convex) in \( \varepsilon \), whereby its integral is convex on the corresponding function space.
Taking the divergence yields:

\[ E^{lin}[u] = \int (C \varepsilon[u] : \varepsilon[u] - fu) \, dx \]
\[ \geq a \int \varepsilon[u] : \varepsilon[u] \, dx - \|f\|_{0,2,D} \|u\|_{0,2,D} \]
\[ \geq a' \|\varepsilon[u]\|_{0,2,D}, \]

so \( E^{lin} \) is bounded from below by 0, thus \( \lim E^{lin}[u^k] =: \bar{E} \) exists, and for \( E^{lin}[u^k] < \bar{E} - \delta \) we get that also \( \|u\|_{1,2,D} \) is bounded.

**Problem (ii)**: Assuming \( C(x) \varepsilon : \varepsilon \geq a|\varepsilon|^2 \) for all \( \varepsilon \in \mathbb{R}^{d,d} \) and all \( x \in D \), we get

\[ E_u = E^{lin}[u] = \int \varepsilon[u] : \varepsilon[u] \, dx - \|f\|_{0,2,D} \|u\|_{0,2,D} \]
\[ \geq a' \|\varepsilon[u]\|_{0,2,D}, \]

so \( E^{lin} \) is bounded from below by 0, thus \( \lim E^{lin}[u^k] =: \bar{E} \) exists, and for \( E^{lin}[u^k] < \bar{E} - \delta \) we get that also \( \|u\|_{1,2,D} \) is bounded.

**Problem 3 (Piola identity)**

Prove the Piola identity \((d = 3):\)

\[ \text{div Cof } D\phi = 0 \]

**Solution**

\[ \text{Cof } A = \text{det } A(A)^{-T} = (\text{adj } A)^T = \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\
 a_{32}a_{13} - a_{33}a_{12} & a_{33}a_{11} - a_{31}a_{12} & a_{31}a_{12} - a_{32}a_{11} \\
 a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix} \]

Counting the indices modulo 3 one gets:

\[ (\text{Cof } D\phi)_{ij} = \phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2 \]

Taking the divergence yields:

\[ \sum_j (\text{Cof } D\phi)_{i,j} = \sum_j (\phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2) = \phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2 + \phi_{i+1,j+1}^2 = 0 \]