Problem sheet 4 10.05.2010

Problem 1 (Existence proof for Lamé-Navier type energies via Lax-Milgram)
Consider the following PDE:
\[-\text{div}\sigma = f\]
Assume \(f \in (H^{1,2})'\), homogeneous Dirichlet boundary values, no Neumann boundary \((\Gamma_F,N)\) and an isotropic material, i.e.
\[
\sigma = \lambda \text{tr} \varepsilon[u] \mathbb{1} + 2\mu \varepsilon[u].
\]
(i) Show that the PDE can be rewritten as
\[-(\lambda + \mu) \nabla \text{div} u - \mu \Delta u = f.\]
(\(\Delta u\) should be understood as by components.)
(ii) Set up a weak formulation for the PDE above.
(iii) Prove existence of a solution directly using the Lax-Milgram theorem. (Do not use Korn’s inequality.)

Problem 2 (Existence proof for linearized elasticity via direct method)
The Lax-Milgram theorem is, where applicable, of course a more convenient method than the direct method in the calculus of variations. Nevertheless, instead of showing the existence of weak solution for the Euler-Lagrange equations, you may also tackle these equations as minimum problems via the direct method:
Consider the elastic energy
\[
E^{\text{lin}}[u] = \int_D (C \varepsilon[u] : \varepsilon[u] - fu) \, dx, \quad u \in H^{1,2}_1(D)
\]
with natural Neumann boundary condition, and a minimizing sequence \((u^k)\).
(i) Show that \(E^{\text{lin}}\) is convex and lower semi-continuous is \(u\) (in fact, \(E^{\text{lin}}\) is continuous, but you will need only lower semi-continuity).
(ii) Using \(A^{(\text{double})}\), show that \((u^k)\) is bounded in \(H^{1,2}_1(D)\) and thus contains a weakly converging subsequence.
(iii) With Mazur’s lemma, argue that there are coefficients $\lambda_i^k \geq 0$, $\sum_i \lambda_i^j = 1$, such that the linear combinations sequence $\sum_i \lambda_i^j u^i$ converges to $\lim u^k$ as $j \to \infty$.

(iv) Of course, wlog you can assume $E[u^k] \leq \lim E[u^k] + \varepsilon$ for any fixed $\varepsilon > 0$. Use the sequence from step (iii), Fatou’s lemma and $E^{lin}$’s lower semicontinuity to show $E^{lin}[u] = \lim E^{lin}[u^k]$.

**Problem 3 (Piola identity)**

Prove the Piola identity ($d = 3$):

$$ \text{div} \ Cof \ D\phi = 0 $$