Problem 1 (Isotropic materials continued)
Based on results from last problem sheet write the stress tensor \( \sigma \) using Voigt notation in terms of \( \nu \) and \( E \) only.

Solution
\[
\sigma = \frac{E}{1+\nu} \left( \frac{\nu}{1-2\nu} \text{tr} \varepsilon \mathbb{1} + \varepsilon \right) = \frac{E}{(1+\nu)(1-2\nu)} \left( \nu \text{tr} \varepsilon \mathbb{1} + (1-2\nu)\varepsilon \right)
\]
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu \\
\nu & 1-\nu & \nu \\
\nu & \nu & 1-\nu \\
1-2\nu & & \\
& 1-2\nu & \\
& & 1-2\nu
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{13} \\
\varepsilon_{23}
\end{bmatrix}
\]

Problem 2 (Stored energy function near a natural state)
Given a homogeneous, isotropic, hyperelastic material, the stored energy function is of the form
\[
W[\phi] = \int_D \overline{W} \left( |D\phi|^2, |\text{Cof} D\phi|^2, (\text{det} D\phi)^2 \right) \, dx.
\]
Assume \( \overline{W} \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \setminus \{0\}) \) and that the reference configuration is a natural state. Prove that the stored energy function can be written as
\[
W[\phi] = \frac{1}{2} \int_D \lambda (\text{tr} \mathcal{E}[u])^2 + 2\mu \text{tr}(\mathcal{E}[u]^2) + o(|\mathcal{E}[u]|^2) \, dx,
\]
where \( \mathcal{E}[u] = \frac{1}{2} (D\phi^T D\phi - 1) \).

Solution
The proof can be done using some results that were not stated explicitly in the lecture:
Theorem 1:
Given a homogeneous, isotropic, elastic material whose reference configuration is a natural state. If the coefficient functions $\gamma_i$ of problem 3 on sheet 2 are differentiable at $i_1$, there are constants such that
\[
\hat{\Sigma}(C) = \lambda (\text{tr} \, \mathcal{E}) \mathbb{1} + 2\mu \mathcal{E} + o(\mathcal{E}), \quad C = \mathbb{1} + 2\mathcal{E}
\]
Proof: see Ciarlet, Theorem 3.7-1., p. 119.

Theorem 2: Assume that $\bar{W}$ is differentiable at a point $i_F = i_C$. The second Piola-Kirchhoff stress tensor is then given by:
\[
\frac{1}{2} \hat{\Sigma}(x, C) = \bar{W}_{\alpha}(\mathbb{1} + \bar{W}_{\beta}(i_1 \mathbb{1} - C) + \bar{W}_{\alpha}t_3C^{-1}, \quad i_k = (i_C)_k
\]
Proof: see Ciarlet, Theorem 4.4-2., p. 153.

Theorem 3: Given a frame-indifferent stored energy function $W(x, A)$, assume that the derivative of $\bar{W}(x, A^T A)$ is a symmetric tensor. The second Piola-Kirchhoff stress tensor can then be written as:
\[
\hat{\Sigma}(x, C) = \bar{W}_{\mathcal{E}}(x, \mathcal{E}), \quad C = \mathbb{1} + 2\mathcal{E}
\]
Proof: see Ciarlet, Theorem 4.2-2., p. 149.

As to the actual problem: Since $\bar{W}$ is twice differentiable, the functions $\bar{W}_{\alpha}$ of theorem 2 are differentiable (at $i_C = i_1$) and therefore also $\hat{\Sigma}(C)$ is differentiable at $i_3$. Thus by theorem 1
\[
\hat{\Sigma}(C) = \lambda (\text{tr} \, \mathcal{E}) \mathbb{1} + 2\mu \mathcal{E} + o(\mathcal{E}).
\]
On the other hand, by theorem 3, $\hat{\Sigma}(C) = \bar{W}_{\mathcal{E}}(\mathcal{E})$. Define a function $\Delta(\mathcal{E})$ by
\[
\Delta(\mathcal{E}) := \bar{W}(\mathcal{E}) - \left( \frac{\lambda}{2} (\text{tr} \, \mathcal{E})^2 + \mu \text{tr} \, \mathcal{E}^2 \right)
\]
In order to differentiate $\Delta(\mathcal{E})$ observe that for a perturbation $H$
\[
\left( \frac{\lambda}{2} (\text{tr} \, (\mathcal{E} + H))^2 + \mu \text{tr} \, (\mathcal{E} + H)^2 \right) = \left( \frac{\lambda}{2} (\text{tr} \, \mathcal{E})^2 + \mu \text{tr} \, \mathcal{E}^2 \right) + \lambda \text{tr} \, \mathcal{E} \text{tr} \,(H) + 2\mu \text{tr} \,(\mathcal{E}H) + o(|H|^2)
\]
\[
= \left( \frac{\lambda}{2} (\text{tr} \, \mathcal{E})^2 + \mu \text{tr} \, \mathcal{E}^2 \right) + (\lambda \text{tr} \,(\mathcal{E}) \mathbb{1} + 2\mu \mathcal{E}) : H + o(|H|^2)
\]
Therefore
\[
\Delta_{\mathcal{E}}(\mathcal{E}) = \bar{W}_{\mathcal{E}}(\mathcal{E}) - \left( \frac{\lambda}{2} (\text{tr} \, \mathcal{E}) \mathbb{1} + 2\mu \mathcal{E} \right)
\]
and thus $\Delta_{\mathcal{E}}(\mathcal{E}) = o(\mathcal{E})$. By the assumption $\Delta$ is continuous differentiable around $\mathcal{E} = 0$ and $\Delta(0) = 0$. Applying Taylor's formula with integral remainder proves the form of the missing remainder.
\[
\Delta(\mathcal{E}) = \int_0^1 \Delta_{\mathcal{E}}(t\mathcal{E}) : \mathcal{E} dt = o(|\mathcal{E}|^2)
\]
Problem 3 (Ogden’s stored energy functions)

Observe that in 3D, a stored energy density $\tilde{W}$ of the form

$$\tilde{W}(A) = a |A|^2 + b |\text{Cof } A|^2 + \Gamma(\text{det } A)$$

for constants $a, b > 0$ and $\Gamma : \mathbb{R} \to \mathbb{R}$ such that $\Gamma(t) \to \infty$ for $t \to 0$ or $t \to \infty$ fulfills the assumptions of the lecture’s remark on $\tilde{W}$’s desired properties.

Name the linear relations between $a, b, \Gamma(1), \Gamma'(1), \Gamma''(1)$ and the Lamé-Navier constants in

$$\tilde{W}_{lin}(E) = \frac{1}{2} (\text{tr } E)^2 + \mu \text{tr } E^2 + o(|E|^2), \quad \Gamma + 2E = A^TA$$

Propose a function $\Gamma$ that increases quadratic for $t \to \infty$ and logarithmic for $t \to 0$.

Solution

Taken from Ciarlet, *Mathematical Elasticity*, pp. 185sq.

Preliminary note: There are several useful relationships between trace and determinant that you should know about. If you do not (up to now), write tr and det in terms of eigenvalues:

1. $2 \text{tr Cof } M = (\text{tr } M)^2 - \text{tr } M^2$
2. $2 \text{det } M = (\text{tr } M)^2 - \text{tr } M^2$ \text{ for } d = 2
3. $6 \text{det } M = (\text{tr } M)^3 - 3 (\text{tr } M)(\text{tr } M^2) + 2 \text{tr } M^3$ \text{ for } d = 3

Compute the linearizations of the terms occurring in $\tilde{W}$:

$$|A|^2 = \text{tr } A^TA = \text{tr } (1 + 2E) = 3 + 2 \text{tr } E \quad \text{tr is linear, tr } 1 = 3 \text{ in } 3D$$

$$|\text{Cof } A|^2 = \text{tr Cof } A^TA \quad \text{cf. lecture, right after Corr. } 1.-7.$$  

$$= \frac{1}{2}(\text{tr } A^TA)^2 - \frac{1}{2} \text{tr}(A^TA)^2 \quad \text{by (3.)}$$

$$= \frac{1}{2}(\text{tr}(1 + 2E))^2 - \frac{1}{2} \text{tr}(1 + 2E)^2$$

$$= 3 + 4 \text{tr } E + 2(\text{tr } E)^2 - 2 \text{tr } E^2$$

$$\text{det } A^TA = \frac{1}{6}(\text{tr } A^TA)^3 - \frac{1}{2}(\text{tr } A^TA)(\text{tr } A^TA)^2 + \frac{1}{4}(\text{tr } A^TA)^3 \quad \text{by (2.)}$$

$$= 1 + 2 \text{tr } E + 2(\text{tr } E)^2 - 2 \text{tr } E^2 + o(|E|^2)$$

$$\Gamma(\text{det } A) = \Gamma((\text{det } A^T)^{1/2})$$

$$= \Gamma(1 + \text{tr } E + \frac{1}{2}(\text{tr } E)^2 - \text{tr } E^2 + o(|E|^2))$$

Taylor: $\sqrt{1+x} = \sqrt{1 + \frac{1}{2}x + o(x)}$

$$= \Gamma(1) + \Gamma'(1)(\text{tr } E + \frac{1}{2}(\text{tr } E)^2 - \text{tr } E^2)$$

Taylor, degree 2

$$+ \frac{1}{2} \Gamma''(1)(\text{tr } E)^2 + o(|E|^2)$$

Compared to the definition of $\tilde{W}_{lin}$, this gives the following system of linear equations:

$$\begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -2 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ \Gamma(1) \\ \Gamma'(1) \\ \Gamma''(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$
As ansatz for $\Gamma$, we take

$$\Gamma(t) := c t^2 - d \log t$$

$$\sim \Gamma'(t) = 2c t - d/t,$$

$$\sim \Gamma''(t) = 2c + d/t^2$$

Verify at your own responsibility that, after choosing a fixed $\Gamma'(1)$,

$$\Gamma''(1) = \lambda + 2\mu + \Gamma'(1)$$

$$a = \mu + \frac{1}{2} \Gamma'(1) \quad b = -\frac{\mu}{2} - \frac{1}{2} \Gamma'(1)$$

$$c = \frac{1}{4} (\Gamma'(1) + \Gamma''(1)) \quad d = \frac{1}{2} (\Gamma''(1) - \Gamma''(1))$$

fulfils the system above. Meaningful solutions will have $a, b > 0$, which reduces the space for $\Gamma'(1)$ to $]-2\mu, -\mu[$, and the request for a convex $\Gamma$ (i.e., a $\Gamma$ with $c, d > 0$) even shrinks the admissible set to $]-\frac{\lambda}{2} - \mu, -\mu[$.

**Problem 4 (Korn’s 2nd inequality, polygonal boundary)**

Give a proof of Korn’s 2nd inequality

$$\|Du\|_D \leq c (\|\varepsilon[u]\|_D + \|u\|_D)$$

for the case of a domain $D \subset \mathbb{R}^2$ with polygonal boundary:

(i) Consider a domain with one edge

$$D = \{(x,y) \mid x > 0, y > \gamma x\}.$$  

Modify the argument from the smooth setting to show that there exists a strain-preserving extension operator to $\mathbb{R}^2 := \{(x,y) \mid x > 0\}$ i.e. an operator $E$ with

$$\|\varepsilon[Eu]\|_{\mathbb{R}^2} \leq c \|\varepsilon[u]\|_D.$$  

(ii) For a polygonal domain, argue with a finite covering which leaves only one edge inside each ball.

**Solution**

In this case ($d = 2$) let $(u, v)$ denote the two components of the displacement (usually denoted by $u$). The extension operator is defined as follows:

$$E(u, v) := \begin{cases} (u, v) & \text{in } D \\ (\tilde{u}, \tilde{v}) & \text{in } \mathbb{R}_+^2 \setminus D \end{cases},$$

where

$$\tilde{u} = pu^\lambda + q v^\mu + r v^\lambda + \sigma v^\mu$$

$$\tilde{v} = ru^\lambda + sv^\mu$$

$$u^\lambda = u(x, \gamma x + \lambda(\gamma x - y))$$

$$v^\mu = v(x, \gamma x + \mu(\gamma x - y)).$$
The extension is clearly defined on the whole domain \( \mathbb{R}^2 \). To ensure continuity the values on the interface \((x, \gamma x)\) have to be checked:

\[
\begin{pmatrix}
  u(x, \gamma x) \\
  v(x, \gamma x)
\end{pmatrix}
\| =
\begin{pmatrix}
  pu(x, \gamma x) + qu(x, \gamma x) + \rho v(x, \gamma x) + \sigma v(x, \gamma x) \\
  rv(x, \gamma x) + sv(x, \gamma x)
\end{pmatrix}
\]

Therefore \( p + q = 1, r + s = 1 \) and \( \rho + \sigma = 0 \) must hold. \((*)\)

In the next step realtions between \( \varepsilon[\tilde{u}, \tilde{v}] \) and \( \varepsilon[u, v] \) need to be established. As in the lecture

\[
\varepsilon_{22}[\tilde{u}, \tilde{v}] = -\lambda \varepsilon_{22}[u, v]^\lambda - \mu \varepsilon_{22}[u, v]^\mu,
\]

leaving \( \varepsilon_{11}[\tilde{u}, \tilde{v}] \) and \( \varepsilon_{12}[\tilde{u}, \tilde{v}] \) to be investigated.

\[
\varepsilon_{11}[\tilde{u}, \tilde{v}] = p(u_x + \gamma(1 + \lambda)u_y)^\lambda + q(u_x + \gamma(1 + \mu)u_y)^\mu + \rho(v_x + \gamma(1 + \lambda)v_y)^\lambda + \sigma(v_x + \gamma(1 + \mu)v_y)^\mu
\]

\[
= p\varepsilon_{11}^\lambda + q\varepsilon_{11}^\mu + \rho\gamma(1 + \lambda)\varepsilon_{22}^\lambda + \sigma\gamma(1 + \mu)\varepsilon_{22}^\mu + 2\rho \underbrace{\varepsilon_{12}^\lambda}_{=\frac{1}{2}(u_x + v_x)} + 2\sigma \underbrace{\varepsilon_{12}^\mu}_{=\frac{1}{2}(u_y + v_y)}
\]

if \( \rho = p\gamma(1 + \lambda) \) and \( \sigma = q\gamma(1 + \lambda) \). \((**)\)

\[
2\varepsilon_{12}[\tilde{u}, \tilde{v}] = n_y + \partial_x
\]

\[
= pu_y^\lambda(-\lambda) + qu_y^\mu(-\mu) + \rho v_y^\lambda(-\lambda) + \sigma v_y^\mu(-\mu) + r(v_x^\lambda + v_y^\lambda\gamma(1 + \lambda)) + s(v_x^\mu + v_y^\mu\gamma(1 + \lambda))
\]

\[
= 2\varepsilon_{12}^\lambda + 2\varepsilon_{12}^\mu + (r\gamma(1 + \lambda) - \lambda p)\varepsilon_{22}^\lambda + (s\gamma(1 + \mu) - \mu\sigma)\varepsilon_{22}^\mu
\]

if \( r = -\lambda p \) and \( s = -\mu q \). \((***)\)

Observe that the parameters \( \lambda = 2, \mu = 1, p = -2, q = 3, r = 4, s = -3, \rho = -6\gamma, \sigma = 6\gamma \) fulfill \((*)\), \((***)\) and \((***)\). Altogether \( \varepsilon[\tilde{u}, \tilde{v}] \) can be expressed as a linear combination of \( \varepsilon[u, v] \) and therefore be estimated:

\[
\|\varepsilon[\tilde{u}, \tilde{v}]\|_{\mathbb{R}^2 \setminus D} \leq c \|\varepsilon[u, v]\|_D
\]

Following the proof in the lecture in case \( g \equiv 0 \) this result can be extended to whole \( \mathbb{R}^2 \).

The last step is the partition of unity argument, almost identical to the one in the lecture. Let \( D \) be a polygonal domain. Then one can choose points \( y_k \) on \( \partial D \) such that each circle \( B_R(y_k) \) either contains a straight line or a domain with one corner. By using translations and rotations the first case was tackled in the lecture while the second was treated above.

The whole proof can be found in: