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DFG-SPP 1114 – Mathematical methods for time series analysis and digital image processing
Morphological methods in 3D Image Fusion and Sequence Analysis in Medical Imaging

Multilevel methods for fast non-rigid matching problems

1 Introduction

Given two images $f_1, f_2 : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$ and $d = 2, 3$, we would like to determine a deformation $\phi : \Omega \to \mathbb{R}^d$ which maps Ω onto Ω and maps grey values in the first image f_1 via a deformation ϕ to grey values at the deformed position in the second image f_2 such that

$$f_1 \circ \phi \approx f_2 \,.$$

We consider ϕ as a perturbation of the identity $\mathbb{I}$ which means $\mathbb{I} + u = \phi$. To optimize the deformation we define the most basic energy E depending on the displacement u:

$$E[u] = \frac{1}{2}\int_\Omega |f_1 \circ (\mathbb{I} + u) - f_2|^2 \,. \tag{E}$$

If u is an ideal deformation the above energy vanishes. Thus we ask for minimizers of the problem $E[u] \to \min$ in some Banach space $\mathcal{V}$. Obviously, this problem is *ill-posed*. Consider a deformation ϕ and for $c \in \mathbb{R}$ the level sets $\mathcal{M}_c^1 = \{x \in \Omega \,|\, f_1(x) = c\}$. Then for any displacement Λ which keeps $\mathcal{M}_c^1$ fixed for all c, the energy does not change, i.e.,

$$E[\phi] = E[\Lambda \circ \phi]$$

This especially holds true for a possible minimizer. Hence, a minimizer – if it exists – is non-unique and the set of minimizers is expected to be *non-regular* and not closed in a usual set of admissible displacements.

2 Regularization

A minimizer u of (E) is characterized by the condition $E'[u] = 0$. In weak formulation:

$$\int_\Omega (f_1 \circ (\mathbb{I} + u) - f_2)\nabla f_1 \circ (\mathbb{I} + u) \cdot \theta = 0 \,,$$

for all $\theta \in [C_0^\infty(\Omega)]^d$. We obtain the L^2-representation of E'

$$\mathrm{grad}_{L^2} E[u] = (f_1 \circ (\mathbb{I} + u) - f_2)\nabla f_1 \circ (\mathbb{I} + u) \,. \tag{1}$$

We investigate a *gradient flow* approach to solve this matching problem. A gradient of a functional $E : \mathcal{V} \to \mathbb{R}$ is defined as the representation of the Fréchet derivative $E' \in \mathcal{V}'$ in a metric $g(\cdot, \cdot)$ on $\mathcal{V}$, i. e.,

$$g(\mathrm{grad}_g E[u], \theta) = \langle E'[u], \theta \rangle \,.$$

Instead of taking a representation in the L^2-product, we introduce a *different length measurement* on the space of deformations and consider a general gradient flow

$$\begin{aligned} \partial_t u &= -\mathrm{grad}_g E[u] \,, \\ u(0) &= u_0 \,. \end{aligned}$$

for a suitable metric $g(\cdot\,, \cdot)$. The choice of $\mathcal{V}$ and the metric g on $\mathcal{V}$ is related to a *regularization* of the matching problem. The representation of the metric g in the duality pairing ($\mathcal{V}'$,$\mathcal{V}$) will be denoted by $A : \mathcal{V} \to \mathcal{V}'$, i.e.,

$$g(\varphi, \psi) = \langle A\,\varphi, \psi \rangle$$

for all $\psi \in \mathcal{V}$. If the inverse of A is regular in a suitable sense the gradient flow can be rewritten as an ODE in the Banach space $\mathcal{V}$:

$$\partial_t u = -A^{-1}E'[u] \,.$$

We can choose $A = \mathbb{I} - \frac{\sigma^2}{2}\Delta$ for $\sigma \in \mathbb{R}^+$. This choice corresponds to an *implicit time discretization of the heat equation* with time-step $\tau = \frac{\sigma^2}{2}$ and is thus related to Gaussian filtering with a filter width σ.

3 A scale–space approach

The energy $E[\cdot]$ is non-convex and we expect an energy landscape with *many local minima*. This implies that gradient descent paths mostly tend to asymptotic states which only locally minimize the energy. We consider a continuous annealing method based on a scale of image pairs $f_{1,\epsilon}$, $f_{2,\epsilon}$, where $\epsilon \geq 0$ is the *scale parameter*. Consider a scale space operator $S(\cdot)$ which maps an initial image f onto some coarser image, i. e.,

$$f_\epsilon = S(\epsilon) f$$

to steer the amount of information in the respective images. We first want to confine to the linear scale space based on Gaussian filtering, i. e. compute

$$\begin{aligned} \partial_t w - \Delta w &= 0 \quad && \text{in } \mathbb{R}^+ \times \Omega \\ \frac{\partial w}{\partial \nu} &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega \\ w(0, \cdot) &= f && \text{in } \Omega \end{aligned}$$

and set $f_\epsilon := w(\frac{\epsilon}{2}, \cdot)$. Alternatively, *other scale space operators* such as *morphological* ones may be incorporated. Now for given $\epsilon \geq 0$ we consider an energy

$$E_\epsilon[u] = \frac{1}{2}\int_\Omega |f_{1,\epsilon} \circ (\mathbb{I} + u) - f_{2,\epsilon}|^2 \,.$$

and the corresponding gradient flow.

Algorithmically, we select a sequence of scales

$$\epsilon_k = \beta_1 2^{-\beta_2 k} \,, \ \beta_1, \beta_2 > 0 \,, \tag{2}$$

and compute discrete counterparts of the continuous solutions $u_{\epsilon_k}(T_k)$ for end times T_k sufficiently large and set

$$u_{\epsilon_k}(0) = u_{\epsilon_{k-1}}(T_{k-1}) \,,$$

For fixed β_2 the parameter β_1 is chosen such that $\epsilon_{k_{max}-1} = h$ and $\epsilon_{k_{max}} = 0$.

4 Discretization

4.1 temporal

We only have to replace the Euclidian distance in $\mathbb{R}^m$ *by the norm induced by* $g(\cdot, \cdot)$ *on* $\mathcal{V}$. We consider the explicit scheme:

$$\frac{u^{n+1} - u^n}{\tau_n} = -A^{-1}E'[u^n] \,.$$

In our implementation we determine τ_n using *Armijo's rule*. Various other strategies *can also be incorporated*.

4.2 Spatial: FE discretization

We suppose $\{\Psi^i\}_{i \in I_h}$ to be the canonical nodal basis of the linear FE space $\mathcal{V}^h$. For given initial displacement U^0 find a sequence of displacements $(U^n)_n$ in $[\mathcal{V}^h]^d$ which solve

$$g_h\left(\frac{U^{n+1} - U^n}{\tau_n}, \Theta\right) = -\langle E'[U^n], \Theta \rangle \,,$$

for all test functions $\Theta \in [\mathcal{V}^h]^d$. Here the metric g_h is a suitable approximation of the original metric g, which leads to the scheme

$$\bar{U}^{n+1} = \bar{U}_n - \tau_n A_h^{-1} \bar{E}'[\bar{U}^n] \,. \tag{3}$$

Here $A_h = M_h G_h$, where M_h is the lumped mass matrix G_h the matrix representation of the discrete metric, i.e.,

$$g_h(X, Y) = M_h G_h \bar{X} \cdot \bar{Y} \,.$$

4.3 Multigrid smoothing

We consider MGM_h (multigrid V-cycle) as an approximation of $\left(\mathbb{I} - \frac{\sigma^2}{2}\Delta_h\right)^{-1}$. For the discrete metric g_h this means $g_h(U, V) = MGM_h^{-1}\bar{U} \cdot \bar{V}$.

4.4 Multilevel structures – coarse scales on coarse grids

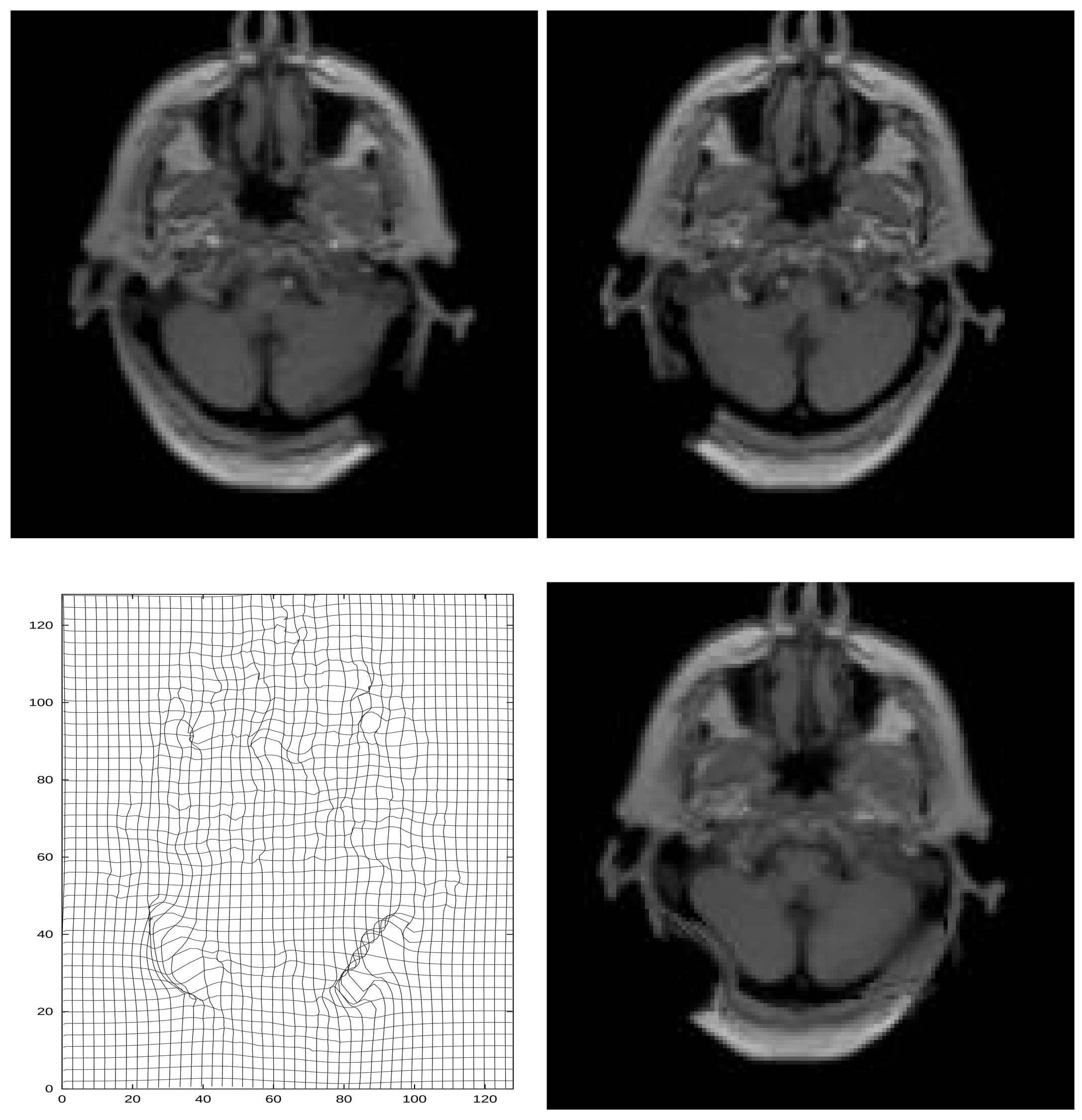
We *couple* a sequence of nested meshes $\mathcal{M}_{h_l}$ which represent the *grid hierarchy* and nested function spaces $\mathcal{V}^l$ with the scales of images. We restrict on scale k the whole problem to grid level $l(k)$. In the case where S corresponds to the heat equation, a suitable choice for $l(k)$ would be the smallest integer such that

$$h_{l(k)} \leq \alpha \epsilon_k$$

for some constant $\alpha \in (0, 1)$.

5 Results

In the first example the matching process has to cope with locally *large deformations*: From top left to bottom right: axial slice through the original 3D image, second image generated by reflection, application of the transformation to a uniform grid, matching result. All 3d examples have a resolution of 129^3 pixels.



Due to the *cascadic multilevel descent* approach and the extensive use of *multigrid smoothing* it has become possible to compute reasonable deformations in less than 5min. Below are the computation times of some elementary components of the algorithm on the finest 129^3 grid.

Process	Duration
V-cycle (single component)	3.3s
computation of $E[u]$ and grad $E[u]$	5.25s
computation of $\langle E'[u], \phi \rangle$	5.38s
computation of $E[u]$	1.23s
time–step control	1-3s

6 Outlook: Morphology

Our aim is to define a matching energy functional, which is *invariant under contrast transformations* of the input images. Those functionals are expected to reveal a much *more robust behaviour* for multi–modal data, or image time sequences with pathological content. Let $\phi : \Omega \to \Omega$ and Gauss-Maps $N_i : \Omega \to \mathbb{R}^d$, $x \mapsto \frac{\nabla f_i(x)}{\|\nabla f_i(x)\|}$ be given. The transformation of normals with respect to the deformation ϕ is then described by

$$N_1 \mapsto \frac{D\phi^{-T} N_1}{\|D\phi^{-T} N_1\|} =: N_1^\phi .$$

We can now formulate a morphological matching energy by

$$E_m(\phi) := \frac{1}{2}\int_\Omega \left\|N_1^\phi - N_2 \circ \phi\right\|^2 ,$$

which depends only on the morphology of the given input images. The minimization of E_m is again an ill–posed problem in the sense, that it wouldn't garuantee a regular unique solution. Thus we have to incorporate another regularizing energy. We plan to use an energy from mathematical elasticity, considering the deformation to be applied on an *hyperelastic material*.

References

[1] U. Clarenz, M. Droske, and M. Rumpf, *Towards fast non–rigid registration*, in Proceedings AMS, 2002.