NONNEGATIVITY PRESERVING CONVERGENT SCHEMES FOR THE THIN FILM EQUATION

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ABSTRACT. We present numerical schemes for fourth order degenerate parabolic equations that arise e.g. in lubrication theory for time evolution of thin films of viscous fluids. We prove convergence and nonnegativity results in arbitrary space dimensions. A proper choice of the discrete mobility enables us to establish discrete counterparts of the essential integral estimates known from the continuous setting. Hence, the numerical cost in each time step reduces to the solution of a linear system involving a sparse matrix. Furthermore, by introducing a time step control that makes use of an explicit formula for the normal velocity of the free boundary we keep the numerical cost for tracing the free boundary low.

0. INTRODUCTION

In this paper we will present new numerical schemes for fourth order degenerate parabolic equations of the form:

$$\begin{array}{lcll} u_t + \operatorname{div}\big(\mathcal{M}(u)\nabla\Delta u\big) & = & 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u & = & 0 & \text{on } \partial\Omega \times (0,T), \\ u(0,\cdot) & = & u_0(\,\cdot\,) & \text{in } \Omega. \end{array} \tag{1}$$

We assume that the nonnegative mobility $\mathcal{M} \in C(\mathbb{R})$ vanishes at zero and that it has at most polynomial growth. By $n := \sup\{s \in \mathbb{R}^+ : \lim_{u\to 0} \frac{\mathcal{M}(u)}{u^s} < \infty\}$, we denote its growth exponent near zero.

Equation (1) models the height of thin films of viscous fluids that – driven by surface tension – spread on plain, solid surfaces. Usually, it is derived by lubrication approximation from the Navier-Stokes equations for incompressible fluids(cf. [3]).
Assuming a *no-slip boundary condition* at the bottom of the thin film, the mobility becomes $\mathcal{M}(u) := |u|^3$, whereas the assumption of various *slip boundary conditions* leads to mobilities of the form $\mathcal{M}(u) = c_1|u|^3 + c_2|u|^\beta$ with positive numbers c_1, c_2 and $\beta \in (0,3)$. Apart from the application in fluid dynamics, degenerate parabolic fourth order equations with a highest order term similar to that in (1) arise in other fields of material sciences. We mention here the Cahn-Hilliard model of phase separation for binary mixtures, where u plays the role of the concentration of one component (cf. [14]), and a plasticity model (cf. [17] and the references therein) where u stands for the density of dislocations.

Crucial for all these applications is the fact, that it is possible to construct solutions of (1) which preserve nonnegativity as has been proved for space dimension $d = 1$ by

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Bernis and Friedman [7] and for higher space dimensions in the papers by Grün [17] and by Elliott, Garcke [14]. This behaviour is in strong contrast to that of classical solutions to linear parabolic equations of fourth order which in general become negative even in the case of strictly positive initial values. Moreover, the publications of Beretta, Bertsch, Dal Passo [2] and of Bertozzi, Pugh [9] who study this equation in space dimension $d = 1$ reveal a rich structure of qualitative behaviour of solutions depending on the mobility growth exponent n. To put it concisely, the larger n is, the stronger is the tendency of solutions to stay positive and the weaker is the regularity at the boundary of the set where u vanishes. In space dimension $d = 1$ for instance, solutions to strictly positive initial data remain strictly positive if $n > \frac{7}{2}$. On the contrary, if $n < \frac{1}{2}$ theoretical results by [2] show that film rupture may occur.
This already indicates that for solutions to (1) maximum or comparison principles cannot be valid. Indeed, all the results about existence and qualitative behaviour mentioned above are the consequence of two basic types of integral estimates, namely the so called energy estimate

$$\int_\Omega |\nabla u(T,.)|^2 + 2\int_{\Omega_T} \mathcal{M}(u)|\nabla\Delta u|^2 = \int_\Omega |\nabla u_0(.)|$$

and the entropy or pressure estimate which reads in its simplest form as

$$\int_\Omega \int_A^{u(T,x)} \int_A^s \frac{1}{\mathcal{M}(r)} dr\, ds\, dx + \int_{\Omega_T} |\Delta u|^2 = \int_\Omega \int_A^{u_0(x)} \int_A^s \frac{1}{\mathcal{M}(r)} dr\, ds\, dx.$$

It is worth mentioning that the number $n = 3$ plays an important role in the theory of equation (1) – both from the mathematical and the physical point of view.
Physically, the assumption of a no-slip condition at the bottom of the thin film which is expressed in the equation by $\mathcal{M}(u) := |u|^3$, leads for spreading droplets to infinite energy dissipation at the triple line solid-gas-liquid(cf. [21] and [13]).
Mathematically, self-similar source type solutions with zero contact angle at the free boundary only exist if $0 < n < 3$ and moreover, all the estimates for proving existence of solutions with zero contact angle break down if $n \geq 3$. (cf. [2], [9] in space dimension 1, [12], [10],and [15] in higher space dimensions).

Another important feature in the qualitative behaviour of solutions to this equation – in particular with regard to applications in wetting and dewetting problems – is the property of having finite speed of propagation. More precisely, this means that the interface separating the regions where u is positive and where u is equal to zero moves with finite velocity as time progresses. For a proof in space dimension $d = 1$, we refer to the work of F. Bernis ([4], [5]), in higher dimensions to [10] and [15].

Those aforementioned issues – nonnegativity of solutions, but lack of comparison principles, propagation of the free boundary – also mean a great challenge in finding efficient numerical schemes.
Just recently, a first successful attempt in constructing a finite element scheme guaranteeing nonnegativity of solutions has been done by Barrett, Blowey and Garcke [1]. By solving in each time-step an elliptic variational inequality of second order, they enforce solutions to stay nonnegative. Unfortunately, it is not clear whether their algorithm guarantees – independently of the grid size – strictly positive discrete solutions in the case

that the continuous solution to be approached is strictly positive.

With this paper, we pursue a different approach by proposing an algorithm discrete in time and space that enables to prove discrete analoga of exactly those integral estimates which are used in the continuous setting for the results about existence and qualitative behaviour. Later on, the discrete analogon of the entropy estimate will be the key to obtain nonnegativity results of discrete solutions for arbitrary $n > 0$ as well as positivity results if $n > 2$ and initial data are strictly positive.
As a byproduct, the numerical cost in each time step reduces to the solution of a linear system involving a sparse band matrix.

The second important issue in numerically simulating wetting phenomena is an efficient tracing of the solution's free boundary.
We will present a new formula – and in fact prove it for self-similar source type solutions – that explicitly expresses the normal velocity of the free boundary in a point $\xi(t)$ in terms of $u(t,\xi(t))$ and of certain spatial derivatives of u in $(t,\xi(t))$.
The discrete counterpart of this formula can easily be implemented to estimate the speed of propagation of the numerical free boundary in each time-step. We use this for an efficient time-step control, defining the time increment at time t by $\tau_c := \frac{h}{speed(t)}$ where h is the grid parameter.
As a consequence, our algorithm reduces the computation time for simulating e.g. the spreading of self-similar source-type solutions by a factor smaller than 0.005.

Let us mention that L. Zhornitskaya and A. Bertozzi(cf. [23]) parallely in time developed a method for proving entropy estimates for numerical schemes that has something in common with our ansatz. They confine themselves to the case $n \geq 2$ and suggest a time-continuous, space-discrete finite difference scheme for approximating strictly positive solutions. For this scheme, they present a proof for strong convergence of positive discrete solutions and show equivalence to a finite element approach on uniform, rectangular grids.

Let us describe the outline of this paper.
In section (1), we will present the finite element scheme to be studied without specifying already at that point the numerical mobilities which we are going to use. Nevertheless, by comparing this finite element method with a finite volume algorithm – both algorithms coincide for certain regular meshes – we will illustrate the main idea how to construct numerical mobilities (or numerical fluxes, respectively) which allow for nonnegative solutions. Section (2) contains a proof of global-in-time existence of discrete solutions by means of a fixpoint argument(Brouwer). In section (3), we prove the discrete analogon to the energy estimate as well as a result about compactness in time. The latter, we will use later on for proving convergence of discrete solutions in higher space dimensions. Section (4) is devoted to a result about uniform discrete Hölder continuity of discrete solutions (using a discrete analogon of $C^{1/2,1/8}(\Omega_T)$) if the dimension is $d = 1$.
In sections (5),(6), the key results of this paper can be found. In section (5), we introduce a general concept of admissible entropy-mobility pairs which allow for discrete analoga of the entropy estimate on arbitrary, unstructured grids. Section (6) is devoted to the proof of nonnegativity results (or positivity results, if $n > 2$) for discrete solutions which are

valid for arbitrary grid size.
Sections (7),(8) contain the convergence results: in space dimension $d = 1$, we get uniform convergence on Ω_T to a solution $u \in C^{1/2,1/8}(\Omega_T)$ satisfying the continuous entropy estimates; in higher space dimensions, strong convergence in the L^2-topology to a function $u \in L^\infty((0,T);H^{1,2}(\Omega)) \cap L^2((0,T);H^2(\Omega))$ is proved.
In section (9), we suggest an explicit formula for the normal velocity of the free boundary in terms of spatial derivatives of u and prove this formula for self-similar source-type solutions in arbitrary space dimensions. Moreover, we present our concept of time-step control.
Finally, section (10) is about numerical experiments in one and two space dimensions. We will discuss phenomena like film rupture, instantaneous development of the zero contact angle in the case of initial data having non-zero contact angle and convergence to solutions of Poisson's problem with constant right-hand side if $n \geq 3$. In particular, we will illustrate the efficiency of the time-step control by comparing explicitly known self-similar source-type solutions with the calculated discrete solutions.

Throughout the whole paper, we use the standard notation for Sobolev spaces, denoting the norm of $W^{k,p}(\Omega)$ $(k \in \mathbb{N}, \quad q \in [1,\infty])$ by $\|\cdot\|_{k,p}$ and abbreviating $W^{k,2}(\Omega)$ and $\|\cdot\|_{k,2}$ by $H^k(\Omega)$ and $\|\cdot\|_k$, respectively. $L^p((0,T);W^{k,q}(\Omega))$ stands for the space of $p-$integrable, measurable functions from the interval $(0,T)$ to $W^{k,q}(\Omega)$. By $(.,.)$, we denote the scalar product in $L^2(\Omega)$, and $(\overline{u})_S$ is an abbreviation for the mean value of u over S.
Finally, $C^{\alpha,\beta}((0,T)\times\Omega)$ stands for the subset of those elements of $C((0,T)\times\Omega)$ which are Hölder continuous to the exponent β (or α) with respect to the first (or second) argument, respectively.

1. Two different numerical approaches

There are two major classes of discretizations for evolution problems, these are finite volume respectively finite element schemes. Here, we consider both types. For simplicity we assume Ω to be polygonally bounded. First we derive a finite volume formulation, which is well suited to motivate the central aim of this paper, how to fix a numerical flux, respectively a numerical mobility, with properties such as mass conservation and guaranteed nonnegativity. Next we compare this approach with a finite element scheme, which will turn out to be preferable concerning the numerical analysis. For such a comparison a duality of the meshes is required. If the finite element mesh consists of open, polygonally bounded subvolumes E, called elements, then a dual mesh is built of open, again polygonally bounded dual cells D_x, corresponding to the vertices x of the primal mesh (cf. figure 1). I. e., we define a single dual cell by

$$D_x := \{y \in \Omega : \mathrm{dist}(y,x) < \mathrm{dist}(y,\tilde{x}),\ \tilde{x} \text{ is vertex of the mesh}\} .$$

For a certain class of meshes, both schemes coincide.
To start with the discussion of finite volume schemes, let us suppose $\Omega = \bigcup_{j\in J} \bar{D}_j$ with open, polygonally bounded cells D_j where J is any index set of finite cardinality and $D_j \cap D_i = \emptyset$ for $i \neq j$.

On each subvolume we can rewrite equation (1) in conservation form

$$\partial_t \int_{D_j} u\,dx \;=\; -\int_{\partial D_j} \mathcal{M}(u)\nabla p\cdot\nu\,dH^{d-1}$$

where $p=\Delta u$ and ν is the outer normal on ∂D_j. The right hand side describes the inflow at the boundary, and $\mathcal{M}(u)\nabla p$ is the corresponding flux.
Let us now discretize in space. That is, we look for $U(t,\cdot),P(t,\cdot)$ piecewise constant on the cells D_j for every $t\geq 0$, such that $\partial_t\int_{D_j} U\,dx$ equals the boundary integral of a numerical flux. Therefore suppose $P=\Delta_h U$, where Δ_h is an appropriate discretization of the Laplace operator. On a regular cell subdivision, we take the standard finite difference discretization with a five point stencil; on an unstructured set of cells, the finite element discrete Laplacian on the corresponding dual mesh is the right choice (see below).

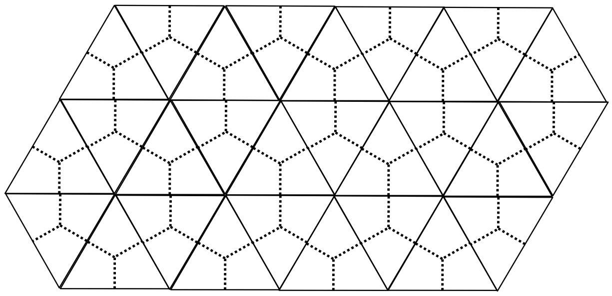

FIGURE 1. A two dimensional finite element triangulation whose edges are outlined in black and the corresponding dual finite volume mesh indicated by dashed lines.

We assume ∇P to be defined uniquely on the whole of Ω, especially on cell faces. On interior faces F of ∂D_j, in general no unique extension of U exists.
To pay account to the fact that the values $U^{\pm}=\lim_{\epsilon\to 0} U(x\pm\epsilon\nu)$ may be different due to the discontinuity of U across cell boundaries, we replace $\mathcal{M}(\cdot)$ by some $M:\mathbb{R}^2\to\mathbb{R}$; $(U^+,U^-)\mapsto M(U^+,U^-)$ and formulate the semi-discrete scheme

$$\partial_t U \;=\; -\,\frac{1}{|D_j|}\int_{\partial D_j} M(U^+,U^-)\nabla p\cdot\nu\,dH^{d-1}\,.$$

For nonempty $F=\bar{D}_j\cap\bar{D}_i$ the inflow on F corresponding to D_j should coincide with the outflow with respect to D_i. Therefore we look for numerical mobilities $M(\cdot,\cdot)$ with the natural property

$$M(U^+,U^-)=M(U^-,U^+)\,.$$

This immediately implies the conservation of mass $\int_\Omega U\,dx$. Major investigations in the following sections will aim at the right choice of the numerical mobility. For the trivial choice $M(U^+,U^-){:=}M(\frac{U^++U^-}{2})$ nonnegativity of the numerical solution can no longer be guaranteed. In section 6 we will be lead to some type of geometric integral mean as the appropriate choice. Finally the semidiscrete scheme can be discretized in time implicitly or explicitly.

Now we turn to a second type of discretizations, the finite element schemes. We denote by $\mathcal{T}_h$ a regular and admissible triangulation of the domain Ω (cf. Ciarlet's monograph [11]). We here restrict ourselves to the case of simplicial grid. Thereby, the triangulation consists of simplicial elements E, i. e. intervals in 1D respectively triangles in 2D, with $\bigcup_{E\in\mathcal{T}_h} E = \Omega$. Here the index h indicates the maximal diameter of an element $E \in \mathcal{T}_h$. Corresponding to $\mathcal{T}_h$ we consider the conforming, linear finite element space $V^h \subset H^{1,2}(\Omega)$. In the following, such discrete functions will be denoted by uppercase letters, in contrast to lowercase letters for arbitrary functions in the nondiscrete function spaces. A function $V \in V^h$ is uniquely defined by its values on the set of nodes $\mathcal{N}_h = \{x_j\}_{j\in J}$ of the triangulation $\mathcal{T}_h$, where J denotes a corresponding index set. To each node x_j corresponds the standard "hat"-type basefunction $\varphi_j \in V^h$ with $\varphi_j(x_i) = \delta_{ij}$. Let us furthermore introduce the well–known lumped masses scalar product corresponding to the integration formula

$$(\Theta, \Psi)_h := \int_\Omega \mathcal{I}_h(\Theta\Psi)$$

where $\mathcal{I}_h : C^0(\Omega) \to V^h$ is the nodal projection operator with $\mathcal{I}_h u = \sum_{j\in J} u(x_j)\varphi_j$.
We recall the following well known estimates:

$$|(U,V) - (U,V)_h| \le Ch^{1+l}\,\|U\|_l\,\|V\|_1 \qquad \text{for all } U,V \in V_h, \quad l = 0,1 \tag{2}$$

In the same spirit, there exist positive constants c, C such that we have for $|.|_h := \sqrt{(.,.)_h}$:

$$c|.|_h^2 \le (.,.) \le C|.|_h^2. \tag{3}$$

A semidiscrete finite element formulation of equation (1) then takes the following form. We look for $(U,P) \in C^1([0,T], V^h \times V^h)$ such that

$$\int_\Omega (\partial_t U, \Theta)_h - (M(U)\nabla P, \nabla\Theta) = 0$$

where $M(U)$ is a numerical mobility and $P = \Delta_h U$. Here $\Delta_h : V^h \to V^h$ denotes the discrete Laplacian with respect to the lumped masses scalar product, whose application to a function $U \in V^h$ is defined by

$$(\Delta_h U, \Psi)_h = -(\nabla U, \nabla\Psi) \quad \forall \Psi \in V^h. \tag{4}$$

The corresponding matrix representation in the nodal basis is $-M_h^{-1}L_h$, where M_h is the diagonal lumped mass matrix with components $(M_h)_{ij} = (\varphi_i, \varphi_j)_h$ and L_h the standard sparse stiffness matrix with entries $(L_h)_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$. Due to the absence of Dirichlet boundary conditions, Δ_h is not injective, i. e. $\ker\Delta_h = \{x \mapsto C | C \in \mathbb{R}\}$. This corresponds to the observation that $\int_\Omega \Delta_h U = 0$, which we immediately see choosing $\Psi \equiv 1$ in (4).

In the above finite element method the replacement of the exact mobility $\mathcal{M}(\cdot)$ by some $M(U)$ can be interpreted as the choice of a specific quadrature to integrate the elliptic term numerically. In our case we suppose that the discrete mobility $M(U)$ for $U \in V^h$ is a symmetric matrix in $\mathbb{R}^{d\times d}$ which is positive semidefinite and piecewise constant on $E \in \mathcal{T}_h$. If $U|_E$ is constant, then $M(U)|_E$ should coincide with $\mathcal{M}(U)\mathrm{Id}$ up to a small perturbation, en detail $M(U)|_E = m(U)\mathrm{Id}$ where $m(u)$ is an appropriate approximation to $\mathcal{M}(u)$ (cf. section 6)). Otherwise $M(U)|_E$ has to be defined appropriately such that again nonnegativity is preserved for U (cf. especially section 5, 6).

Let us underline the close relation between the choice of $M(U^+, U^-)$ in the finite volume context and $M(U)|_E$ in case of a finite element scheme remarking that both schemes coincide for a certain type of triangulation $\mathcal{T}_h$ and corresponding dual cell subdivision $\{D_j\}_{j\in J}$. Therefore we consider $\Omega = [0,1]^2$ and a regular grid of points $\{q_{ij}\}_{0\le i,j\le N}$

where $q_{ij}:=(\frac{i}{N},\frac{j}{N})$. Then for every index pair (i,j), $0 \leq i,j \leq N-1$, we define triangular elements E, respectively E' by the set of vertices $\{q_{ij}, q_{i+1,j}, q_{i,j+1}\}$, respectively $\{q_{i+1,j}, q_{i+1,j+1}, q_{i,j+1}\}$. Now we consider as finite element triangulation $\mathcal{T}_h$ the set of all these elements, and as the finite volume mesh the set of corresponding dual cells. Finally we define $M(U^+,U^-)$ and $M(U)|_E$ according to the definition in section 6. Then a tedious but straightforward computation which we skip here proves the equivalence of the different approaches.

In what follows, we will focus on the finite element discretization. Let us discretize now the above semidiscrete scheme in time. Therefore suppose $[0,T]$ to be subdivided in intervals $I_k = (t_k, t_{k+1}]$ with $t_{k+1} = t_k + \tau_k$ for time increments $\tau_k > 0$ and $k = 0,\cdots,N-1$. We will use forward and backward difference quotients with respect to time which we shall henceforward denote by ∂_τ^+ or ∂_τ^-, respectively. Now we can formulate an fully implicit, backward Euler discretization scheme for equation (1):

For given $U^0 \in V^h$ find a sequence (U^k, P^k) for $k = 0,\cdots,N-1$ with $U^k, P^k \in V^h$ such that

$$\left(\partial_\tau^- U^{k+1}, \Theta\right)_h - \left(M(U^{k+1})\nabla P^{k+1}, \nabla\Theta\right) = 0 \qquad (5)$$

$$\left(P^{k+1}, \Psi\right)_h = -\left(\nabla U^{k+1}, \nabla\Psi\right) \qquad (6)$$

for all $\Theta, \Psi \in V^h$.
Choosing $\Psi \equiv 1$ in (4), we immediately observe that

$$\int_\Omega P^{k+1} = 0\,.$$

The discrete initial values U^0 are assumed to be an approximation of the continuous initial values u_0. Suppose $u_0 \in C^0$, then we can prescribe $U^0 := \mathcal{I}_h u_0$.
Furthermore, we introduce $S^{0,-1}(V^h)$ as the space of functions $V : [0,T] \to V^h$ which are piecewise constant in time on the intervals I_k and with $V(t) \in V^h$ for all $t \in [0,T]$. For the discrete solutions U^k, P^k corresponding to the sequence of timesteps $\{t_k\}_{k=1,\cdots,N}$, we then straightforward define a piecewise constant extension $U_{\tau h} \in S^{0,-1}(V^h)$ in time by $U_{\tau h}(t) := U^k$ for $t \in I_k$. Furthermore

$$\tilde{U}_{\tau h}(t) := \frac{t_{k+1}-t}{\tau_k} U^k + \frac{t-t_k}{\tau_k} U^{k+1}$$

represents a linear and continuous interpolation in time in the corresponding function space which we denote by $S^{1,0}(V^h)$. A pressure $P_{\tau h} \in S^{0,-1}(V^h)$ and a continuous pressure $\tilde{P}_{\tau h} \in S^{1,0}(V^h)$ can be defined by analogy. In particular, $P_{\tau h} = \Delta_h U_{\tau h}$, respectively $\tilde{P}_{\tau h} = \Delta_h \tilde{U}_{\tau h}$. We will call a pair $(U_{\tau h}, P_{\tau h})$, that solves the equations (5), (6) with initial condition $U^0 = \mathcal{I}_h u_0$, a discrete solution. To simplify the writing we will skip the indices whenever a misunderstanding is ruled out by the context.

2. Existence of discrete solutions

In this section, we will prove the existence of discrete solutions globally in time by use of a fixpoint argument. For $W^k = U^k - \alpha$ with $\alpha := \frac{1}{|\Omega|}\int_\Omega U^0$ we obtain the weak equations

$$\left(\partial_\tau^- W^{k+1}, \Theta\right)_h - \left(M(W^{k+1}+\alpha)\nabla P^{k+1}, \nabla\Theta\right) = 0 \qquad (7)$$

$$\left(P^{k+1}, \Psi\right)_h = -\left(\nabla W^{k+1}, \nabla\Psi\right)$$

and the initial condition $W^0 = U^0 - \alpha$. At first let us define the weighted stiffness matrix $L_h(W)$ for $W \in V^h$ by

$$(L_h(W)_{ij})_{i,j\in J} \;:=\; \int_\Omega M(W+\alpha)\nabla\varphi_i \cdot \nabla\varphi_j\,.$$

Then a solution of (5) is obtained solving the following nonlinear system of $q = \dim V^h$ equations for each time step. If we denote the nodal value vector for a function $V \in V^h$ by $\bar V$, and with a slight misuse of notation rewrite $L_h(\bar W)$ for $L_h(W)$, then for given $\bar W^k \in \mathbb{R}^q$ we search $\bar W^{k+1} \in \mathbb{R}^q$ such that $F(\bar W^{k+1}) = 0$ for

$$F(\bar W) \;=\; \left(\mathrm{Id} + \tau_k M_h^{-1} L_h(\bar W) M_h^{-1} L_h\right)\bar W - \bar W^k$$

Let us now introduce a new bilinear form on $\mathbb{R}^q$ by

$$<\bar W, \bar V> := L_h \bar W \cdot \bar V$$

where $\cdot$ indicates the Euclidian scalar product on $\mathbb{R}^q$. By definition this form is symmetric and therefore a scalar product on $K^\perp := \left\{\bar W \mid M_h \bar W \cdot (1,\cdots,1) = 0\right\}$. We easily verify that $\bar W^0 \in K^\perp$ and by induction that $F : K^\perp \to K^\perp$. Furthermore considering especially the assumptions on $M(U)$ we estimate

$$\begin{aligned} <F(\bar W), \bar W> \;&=\; <\bar W - \bar W^k, \bar W> + \tau_k L_h(\bar W) A_h \bar W \cdot A_h \bar W \\ &\geq\; <\bar W - \bar W^k, \bar W> \geq 0 \end{aligned}$$

for $<\bar W, \bar W>^{\frac12} \geq R$ with $R := <\bar W^k, \bar W^k>^{\frac12}$. Therefore we can apply Brouwer's fixpoint theorem and prove existence of a root $\bar W^{k+1}$ for the mapping $F(\cdot)$. Finally we define by $U^{k+1} := (W^{k+1} + \alpha)$ a solution of the original problem.
Let us remark that the restriction on $K^\perp$ reflects the mass conservation property $\int_\Omega U^{k+1} = \int_\Omega U^0$, which we immediately obtain choosing $\Theta \equiv 1$ in (5). It is conserved by typical iterative solvers, such as Newton's method or nonlinear Gauss–Seidel iterations.

3. Basic a priori estimates

Main topic of this section is the derivation of a priori estimates necessary for compactness results of a sequence of discrete solutions. It turns out that these are the discrete counterparts of analoguous estimates in the continuous theory. In what follows we assume fixed timesteps $\tau_k = \tau = \frac{T}{N}$ for $N \in \mathbb{N}$ to simplify the presentation. Let us start with an energy type estimate.

Lemma 3.1. *(Energy estimate)*
Let $(U_{\tau h}, P_{\tau h}) \in S^{0,-1}(V^h) \times S^{0,-1}(V^h)$ be a discrete solution.
Then the following a priori estimate holds:

$$\begin{aligned} \frac{1}{2}\int_\Omega \left|\nabla U^N(x)\right|^2 dx \quad &+ \quad \frac{1}{2}\sum_{i=0}^{N-1}\int_\Omega \left|\nabla(U^{i+1}(x) - U^i(x))\right|^2 dx \\ &+ \quad \int_0^T\int_\Omega M(U)\left|\nabla P\right|^2 dxdt = \frac{1}{2}\int_\Omega \left|\nabla U^0(x)\right|^2 dx. \end{aligned}$$

In particular, if $U^0_{\tau h}$ is uniformly bounded in $H^1(\Omega)$, then $U_{\tau h}$ and $M(U)\left|\nabla P\right|^2$ are uniformly bounded in $L^\infty((0,T); H^1(\Omega))$ or $L^1((0,T)\times\Omega))$, respectively, by a constant C that is independent of τ, h.

Proof. We choose $\Theta = P^{k+1}$ in equation (5), and summing over k, we obtain for the parabolic part with the help of equation (6)

$$\begin{aligned} \frac{1}{\tau}\sum_{k=0}^{N-1}\left(U^{k+1} - U^k, P^{k+1}\right)_h \quad &= \quad \frac{1}{\tau}\sum_{k=0}^{N-1}\left(\nabla U^{k+1} - \nabla U^k, \nabla U^{k+1}\right) \\ &= \quad \frac{1}{2\tau}\sum_{k=0}^{N-1}\int_\Omega \left|\nabla U^{k+1}\right|^2 + \left|\nabla U^{k+1} - \nabla U^k\right|^2 - \left|\nabla U^k\right|^2 dx \end{aligned}$$

and for the elliptic term

$$\sum_{k=0}^{N-1}\left(M(U^{k+1})\nabla P^{k+1}, \nabla P^{k+1}\right) \quad = \quad \frac{1}{\tau}\int_0^T (M(U)\nabla P, \nabla P)$$

Hence, multiplying by τ, the stated estimate is established. If we furthermore consider the mass conservation $\int_\Omega U^k = \int_\Omega U^0$, we obtain as a straightforward consequence uniform bounds for the inspected norms. □

In space dimension $d = 1$, Sobolev's imbedding result immediately gives the following corollary which will be the starting point for the proof of uniform discrete Hölder-continuity of numerical solutions $U_{\tau h}$ in section (4) and later on of their uniform convergence.

Corollary 3.2. *Let $(U_{\tau h}, P_{\tau h}) \in S^{0,-1}(V^h) \times S^{0,-1}(V^h)$ be a discrete solution and suppose that $\|U^0_{\tau h}\|_1$ is uniformly bounded in τ, h.*
Then $U_{\tau h}$ is uniformly bounded in $L^\infty((0,T); C^{1/2}(\Omega))$ for τ, h tending to 0.

Let us consider now a result on compactness in time valid in any space dimension. Combined with the energy estimate, it will be sufficient to prove the existence of convergent subsequences.

Lemma 3.3. *(Compactness in time)*
Let $(U_{\tau h}, P_{\tau h})$ be a discrete solution and let $s < T$ be a positive number. Further, assume the existence of a constant M_1 such that

$$\max_{1\le i,j\le d}\ \sup_{(t,x)\in[0,T]\times\Omega} M_{ij}(U_{\tau h})(t,x) \le M_1. \tag{8}$$

Then, there exists a constant $C > 0$ such that

$$\int_0^{T-s} \Big(U(t+s,x) - U(t,x), U(t+s,x) - U(t,x) \Big)_h dt \le C M_1 s \tag{9}$$

Proof. Let us first prove the result for values $s = l\tau$, $l \le N$ a positive integer. For a fixed number j satisfying $0 \le j \le N - l$ we choose

$$\Theta := U^{j+l} - U^j$$

in equation (5), multiply by τ and sum over $k = j-1, \cdots, j+l-1$. This implies:

$$\tau \sum_{k=j-1}^{j+l-1} \left(\frac{U^{k+1} - U^k}{\tau}, U^{j+l} - U^j \right)_h = \tau \sum_{k=j-1}^{j+l-1} \int_\Omega M(U^{k+1}) \nabla P^{k+1} \nabla (U^{j+l} - U^j) \tag{10}$$

As the term on the left-hand side is equal to $(U^{j+l} - U^j, U^{j+l} - U^j)_h$, it follows:

$$\big(U^{j+l} - U^j, U^{j+l} - U^j\big)_h \le M_1 \tau \sum_{k=0}^{l} \left(\int_\Omega M(U^{j+k}) \left|\nabla P^{j+k}\right|^2 \right)^{\frac{1}{2}} \left(\int_\Omega \left|\nabla (U^{j+l} - U^j)\right|^2 \right)^{\frac{1}{2}} \tag{11}$$

Now, we sum up from $j = 1$ to $N - l$ and apply the energy estimate (cf. equation (8)). Thus, we obtain

$$\begin{aligned}
&\tau \sum_{j=1}^{N-l} \big(U^{j+l} - U^j, U^{j+l} - U^j\big)_h \\
&\le M_1 \tau \sum_{k=0}^{l} \left(\tau \sum_{j=1}^{N-l} \int_\Omega M(U^{j+k}) \left|\nabla P^{j+k}\right|^2 \right)^{\frac{1}{2}} \left(\tau \sum_{j=1}^{N-l} (\int_\Omega \left|\nabla (U^{j+l} - U^j)\right|^2 \right)^{\frac{1}{2}} \\
&\le C M_1 l \tau ,
\end{aligned}$$

which is inequality (9) for $s = l\tau$. For arbitrary $0 < s < T$, $s = r\tau$ with $0 < r < N$, we argue as follows: Writing $r = l + \theta$ with $\theta \in (0,1)$ and l a nonnegative integer, we get:

$$U(t + r\tau, x) = \begin{cases} U(t + l\tau, x) & \text{if } t \in (j\tau, j\tau + (1-\theta)\tau] \\ U(t + (l+1)\tau) & \text{if } t \in (j\tau + (1-\theta)\tau, (j+1)\tau] \end{cases} \tag{12}$$

With the notation

$$\Psi_{lj} := \Big(U((j+l)\tau, .) - U(j\tau, .), U((j+l)\tau, .) - U(j\tau, .) \Big)_h \qquad j = 0, ..., N-l,$$

we obtain using $U \in S^{0,-1}(V_h)$:

$$\begin{aligned}
&\int_0^{T-r\tau} \Big(U(t + r\tau, x) - U(t,x), U(t + r\tau, x) - U(t,x) \Big)_h dt \\
&= \sum_{j=0}^{N-l-1} (1-\theta) \Psi_{lj} + \sum_{j=0}^{N-l-1} \theta \Psi_{l+1,j} \\
&\le C M_1 ((1-\theta) l + \theta (l+1)) \tau = C M_1 r \tau = C M_1 s
\end{aligned}$$

This proves the lemma. □

4. Uniform Hölder continuity

The result on uniform discrete Hölder continuity of discrete solutions basically relies on two facts, first the uniform $L^\infty((0,T);C^{1/2}(\Omega))$-regularity established in corollary 3.2, and secondly the following lemma about Hölder continuity in time of the spatial mean.

Lemma 4.1. *Suppose $d=1$ and $(U_{\tau h},P_{\tau h})$ is a discrete solution. Furthermore assume that $M(\cdot)$ is bounded by a constant M_1. Then, for integers $k\geq 0$, $l>0$ with $k+l\leq N$, the following estimate is valid independently of h:*

$$\frac{1}{2}\Big(U^{j+l}-U^j,U^{j+l}-U^j\Big)_h\leq M_1\left\|U^0\right\|_1\sqrt{l\tau} \tag{13}$$

Remark: In section (6), it will be verified that the boundedness condition on $M(\cdot)$ is always satisfied if $d=1$.

Proof. We choose $\Theta=U^{k+1}-U^j$ in equation (5), multiply by τ and sum over $k=j,\cdots,j+l-1$. Then we obtain for the parabolic term in analogy to the proof of lemma 3.1

$$\sum_{k=j}^{j+l-1}\left(U^{k+1}-U^k,U^{k+1}-U^j\right)_h=\sum_{k=j}^{j+l-1}\left((U^{k+1}-U^j)-(U^k-U^j),U^{k+1}-U^j\right)_h$$
$$=\frac{1}{2}\left(U^{j+l}-U^j,U^{j+l}-U^j\right)_h+\frac{1}{2}\sum_{k=j}^{j+l-1}\left(U^{k+1}-U^k,U^{k+1}-U^k\right)_h$$

and taking into account the energy estimate, the elliptic part can be estimated as follows:

$$\sum_{k=j}^{j+l-1}\tau\left(M(U^{k+1})\nabla P^{k+1},\nabla(U^{k+1}-U^j)\right)$$
$$=\left|\int_{j\tau}^{(j+l)\tau}\int_\Omega M(U)\nabla P\nabla(U-U^j)\right|$$
$$\leq\sqrt{M_1}\int_{j\tau}^{(j+l)\tau}\left(\int_\Omega M(U)|\nabla P|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega\nabla(U-U_j)^2\right)^{\frac{1}{2}}$$
$$\leq\sqrt{M_1}\max_{k=j,\cdots j+l}\left\|U^k\right\|_1\left(\int_{j\tau}^{(j+l)\tau}\int_\Omega M(U)|\nabla P|^2dxdt\right)^{\frac{1}{2}}\left(\int_{j\tau}^{(j+l)\tau}1\,dt\right)^{\frac{1}{2}}$$
$$\leq\sqrt{M_1}\left\|U^0\right\|_1\sqrt{l\tau}$$

Hence, the assertion follows. □

Now, we are in the position to prove the main result of this section – adapting an idea of F. Otto [18] to the discrete setting:

Lemma 4.2. *Assume $d=1$ and that for integer $l,k>0$ with $l+k<N$ the relation $k\tau\geq h^4$ holds. Then for a discrete solution $(U_{\tau h},P_{\tau h})$ with $\|M(U_{\tau h})\|_\infty\leq M_1$ independently of τ,h, there exists a constant C depending only on $\|U^0\|_1$ such that*

$$|U^{l+k}(x)-U^l(x)|\leq C\,(k\tau)^{\frac{1}{8}} \tag{14}$$

for $x\in\Omega$.

Remark: In section (9) we will prove that $\max_{\tau_c,h\to 0}(\frac{h^4}{\tau_c}) \to 0$ for the time increment τ_c given by the time-step control provided $n \geq 1$. Hence, for those mobilities the compatibility condition for τ, h expressed in the lemma, does no longer mean any restriction.

Proof. For given $x \in \Omega$ and a small positive number δ, let us assume without loss of generality that $[x, x+\delta) \subset \Omega$. Then we calculate

$$\begin{aligned} |U^{l+k}(x) - U^l(x)| &= \Big| \fint_x^{x+\delta} U^{l+k}(x) - U^{l+k}(y)\, dx + \fint_x^{x+\delta} U^{l+k}(y) - U^l(y)\, dy + \\ &\quad \fint_x^{x+\delta} U^l(y) - U^l(x)\, dy \Big| \\ &= |\, I + II + III \,| \end{aligned}$$

Corollary (3.2) and Cauchy Schwarz's inequality imply

$$\begin{aligned} |\, I + III \,| &\leq \delta^{\frac{1}{2}} \left(\left\| U^{l+k} \right\|_1 + \left\| U^l \right\|_1 \right) \\ &\leq 2\delta^{\frac{1}{2}} \left\| U^0 \right\|_1 . \end{aligned}$$

For the second term, we apply lemma 4.1 and achieve

$$\begin{aligned} |\, II \,| &\leq \delta^{-\frac{1}{2}} \left(\int_\Omega (U^{l+k} - U^l)^2 \right)^{\frac{1}{2}} \\ &\leq \delta^{-\frac{1}{2}} \left(h \left(\left\| U^{l+k} \right\|_1 + \left\| U^l \right\|_1 \right) + \left(U^{l+k} - U^l, U^{l+k} - U^l \right)_h^{\frac{1}{2}} \right) \\ &\leq \delta^{-\frac{1}{2}} \left(2h \left\| U^0 \right\|_1 + M_1 (k\tau)^{\frac{1}{4}} \left\| U^0 \right\|_1 \right) . \end{aligned}$$

Here, we have applied inequality (2) to estimate the difference between the L^2-norm and the norm corresponding to the lumped masses scalar product. Finally, we choose $\delta = (k\tau)^{\frac{1}{4}}$, take into account $h < (k\tau)^{\frac{1}{4}}$ and end up with

$$|\, I + II + III \,| \leq C \left(\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}} (k\tau)^{\frac{1}{4}} \right) \leq C (k\tau)^{\frac{1}{8}} .$$

□

5. Pressure-entropy a priori estimates

Now we will present an abstract frame which enables us to prove a discrete counterpart of the continuous integral estimate

$$\int_\Omega \int_A^{u(T,x)} \int_A^s \frac{1}{\mathcal{M}(r)} dr\, ds\, dx + \int_{\Omega_T} |\Delta u|^2 = \int_\Omega \int_A^{u_0(x)} \int_A^s \frac{1}{\mathcal{M}(r)} dr\, ds\, dx.$$

This estimate is sometimes called entropy estimate and serves for the proof of nonnegativity results in the continuous setting.

Let us start with some notation. By $m : \mathbb{R} \to \mathbb{R}_0^+$, we denote an approximation of the continuous mobility $\mathcal{M}$ that will be specified later on (cf. the following section). A is an arbitrary, but fixed positive number.

We call a pair of functions $G : \mathbb{R} \to \mathbb{R}_0^+$, $M : V^h \to \mathbb{R}^{d\times d}$ an *admissible entropy-mobility pair with respect to the triangulation* $\mathcal{T}_h$ if the following axioms are satisfied:

(i) $M : V^h \to \mathbb{R}^{d\times d}$, $M(U)$ is piecewise constant on every $E \in \mathcal{T}_h$,
(ii) M is continuous, $M(U)|_E = m(U)\mathrm{Id}$ if $U|_E$ is constant,
(iii) $M^T(U)\nabla\mathcal{I}_h G'(U) = \nabla U$, where $G(s) := \int_A^s g(r)dr$ with $g(s) = \int_A^s m(r)^{-1}dr$,
(iv) $M(U)$ is symmetric and positive semidefinite.

For further reference, let us remark that G is nonnegative and convex by construction.

At first, we consider elements whose faces form right angles at one vertex. Let us assume that $\hat{E} = \hat{E}_{(\alpha_1,\cdots,\alpha_d)}$ is a reference simplex in $\mathbb{R}^d$ with corners $x_0 = 0$, $x_i = \alpha_i e_i$ for $i = 1,\cdots,d$ and $\alpha_i \in \mathbb{R}$, where e_i denotes the ith unit vector. Applying the notation $U_i = U(x_i)$ and $g = G'$, we now look for a matrix $\hat{M}$ on $\hat{E}$ with $\hat{M}^T\nabla_{\hat{x}}\mathcal{I}_h g(U) = \nabla_{\hat{x}}U$, where $\nabla_{\hat{x}}$ is the gradient on $\hat{E}$. Due to

$$g(U_i) - g(U_0) \;=\; \int_{U_0}^{U_i} \frac{1}{m(s)}\,ds$$

we straightforward verify that

$$\hat{M} = \left(\hat{M}_{ij}\right)_{i,j=1,\cdots,d} \qquad \text{with } \hat{M}_{ij} = \left(\fint_{U_0}^{U_i} \frac{1}{m(s)}\,ds\right)^{-1}\delta_{ij}$$

satisfies our axioms above. For $U_k = U_0$ the definition simplifies to $\hat{M}_{kk} = m(U_0)$.
Now we generalize this method to arbitrary elements E which have a vertex x_0 with the property that any two edges intersecting each other in x_0 form a right angle. We can find $(\alpha_1,\cdots,\alpha_d)$ and an orthogonal matrix A in such a way that the affine linear mapping $\hat{x} \mapsto x = x_0 + A\hat{x}$ is a bijection between the reference element $\hat{E}_{(\alpha_1,\cdots,\alpha_d)}$ and E. We proceed with the pull back on $\hat{E}_{(\alpha_1,\cdots,\alpha_d)}$ and obtain $A^T\nabla_x\varphi(x) = \nabla_{\hat{x}}\varphi(\hat{x})$. This implies

$$\begin{aligned} \hat{M}^T\nabla_{\hat{x}}\mathcal{I}_h g(U) &= \nabla_{\hat{x}}U \quad \Rightarrow \\ \hat{M}^T A^T\nabla_x\mathcal{I}_h g(U) &= A^T\nabla_x U \quad \Rightarrow \\ \left(A^{-T}\hat{M}^T A^T\right)\nabla_x\mathcal{I}_h g(U) = \nabla_x U\,. & \end{aligned}$$

Therefore defining $M{:=}A\hat{M}A^{-1}$, the conditions (ii), (iii) are fulfilled on E. Since A is orthogonal, M is symmetric and positive semidefinite; hence condition (iv) is satisfied, too.

In the general case we cannot argue vertex oriented. Nevertheless, we can define

$$M_{ij}{:=}\frac{\partial_{x_i}U}{\partial_{x_i}\mathcal{I}_h g(U)}\delta_{ij}. \tag{15}$$

For an efficient implementation, this definition is less constructive than the first one which we therefore find preferable. Finally using this method on every $E \in \mathcal{T}_h$, the required property (i) holds and we obtain an admissible entropy–mobility pair (G, M) .

Let us emphasize that the choice $\Theta = \mathcal{I}_h G'(U)$ implies for the elliptic term from (5) the identity

$$-\int_\Omega M(U)\nabla P\nabla\Theta = -\int_\Omega M(U)\nabla P\nabla\mathcal{I}_h G'(U) = -\int_\Omega \nabla P\nabla U = (P,P)_h\ . \tag{16}$$

After these preliminaries, we can prove the key lemma for the subsequent results on regularity and nonnegativity of solutions. It reads as follows:

Lemma 5.1. *(Pressure-Entropy estimate)*
Let (U,P) be a solution to the system of equations (5)-(6) and assume that (M,G) is an admissible entropy-mobility pair as described above.
Then, for arbitrary $T = K\tau$, $K \in \mathbb{N}$, the following estimate holds:

$$\begin{aligned}\int_\Omega \mathcal{I}_h G(U(T,x))dx + &\int_\tau^T \Big(P(t,.),P(t,.)\Big)_h dt \\ &\le \int_\Omega \mathcal{I}_h G(U^0(x))dx\end{aligned} \tag{17}$$

Proof. We take the function $\mathcal{I}_h G'(U^{k+1})$ as test function in the weak formulation (5) and obtain:

$$\Big(\partial_\tau^- U^{k+1}, \mathcal{I}_h G'(U^{k+1})\Big)_h - \int_\Omega M(U^{k+1})\nabla P^{k+1}\nabla \mathcal{I}_h G'(U^{k+1})dx = 0 \tag{18}$$

Using property (iv) of admissible entropy-mobility pairs (G,M) as well as identity (16), we get:

$$\Big(\partial_\tau^- U^{k+1}, \mathcal{I}_h G'(U^{k+1})\Big)_h + \Big(P^{k+1}, P^{k+1}\Big)_h = 0 \tag{19}$$

The convexity of G implies:

$$\frac{1}{\tau}(G(U^{k+1}(x)) - G(U^k(x))) \le \partial_\tau^- U^{k+1}(x) G'(U^{k+1}(x)) \tag{20}$$

Hence, we can estimate:

$$\Big(\mathcal{I}_h(G(U^{k+1}(x)) - G(U^k(x))), 1\Big)_h + \Big(P^{k+1}, P^{k+1}\Big)_h \le 0 \tag{21}$$

Summing up from $k=0$ to $K-1$, multiplying by τ and using the fact that $(\mathcal{I}_h\eta, 1)_h = \int_\Omega \mathcal{I}_h \eta(x)dx$, we obtain the result. □

6. Nonnegativity results for discrete solutions

In this section, we will be concerned with the explicit construction of discrete mobilities M that allow for nonnegative discrete solutions U. By a slight misuse of semantics, the adjective *nonnegative* means in this context that it is possible to find mobilities M_σ depending on a positive control parameter σ such that for given $\varepsilon > 0$ the corresponding discrete solution $U^\sigma_{\tau h}$ satisfies $U^\sigma_{\tau h} \ge -\varepsilon$ on Ω_T. If the growth coefficient n is greater 2, even strict positivity results can be proved for discrete solutions provided initial data are strictly positive. These results are exactly the discrete counterpart of the nonnegativity results which follow in the continuous setting from the basic entropy estimates(cf. [7] and [17]).
Precisely, the following theorem holds:

Theorem 6.1. *(Existence of nonnegative discrete solutions $U^\sigma_{\tau h}$)*
Let $\mathcal{T}_h$ be an admissible triangulation of Ω and let $n > 0$ be the growth coefficient of $\mathcal{M}$ in zero. Assume that the mobility $\mathcal{M}$ is monotoneously increasing and vanishes on $\mathbb{R}_- \cup \{0\}$. For arbitrary $\varepsilon > 0$, there exists a positive control parameter σ_0 which only depends on d, n, ε, h and the initial datum $u_0 \ge 0$ such that:
For every $0 < \sigma < \sigma_0$ discrete entropy-mobility pairs (G_σ, M_σ) can be constructed having the property that the corresponding discrete solutions $U^\sigma_{\tau h}$ of equation (5) satisfy:

- $U^{\sigma}_{\tau h} > -\varepsilon$ *if* $u^0 \geq 0$ *and* $0 < n < 2$.
- $U^{\sigma}_{\tau h} > -\varepsilon$ *if* $u^0 \geq \sigma_0$ *and* $n = 2$.
- $U^{\sigma}_{\tau h} > \sigma/2$ *if* $u^0 \geq \sigma_0$ *and* $n > 2$.

Proof. For the ease of presentation, we will assume at the beginning that $d = 1$. Later on, the modifications to be applied in the higher spatial dimensional case will be straightforward.
We will construct entropy-mobility pairs (G_σ, M_σ) in two different ways depending on whether the mobility growth coefficient n is smaller than 1 or not.
If $n \geq 1$, we start with the following shifted mobilities $m_\sigma(u) := \mathcal{M}(\max(\sigma, u))$, calculate the entropy G_σ as the second primitive of m_σ^{-1} and eventually define the corresponding discrete mobility $M_\sigma(U)|_E$ on an element E of the triangulation $\mathcal{T}_h$ by the formula:

$$M_\sigma(U)\Big|_E := \begin{cases} m_\sigma(U_1) & \text{if } U_1 = U_2 \\ \left(⨍_{U_1}^{U_2} \frac{1}{m_\sigma(s)} ds \right)^{-1} & \text{if } U_1 \neq U_2 \end{cases} \tag{22}$$

Here, U_1, U_2 denote the values of U on the boundary of E.

In the case $0 < n < 1$, we obtain the discrete entropy G_σ by an C^1-extension of an appropriate continuous entropy G to the negative numbers and successively define $M_\sigma|_E$ by the formula

$$M_\sigma(U)\Big|_E := \begin{cases} (U_1 - U_2)(G'_\sigma(U_1) - G'_\sigma(U_2))^{-1} & \text{if } U_1 \neq U_2 \\ \mathcal{M}(U_1) & \text{if } U_1 = U_2 \geq 0 \\ \sigma \mathcal{M}(-U_1) & \text{if } U_1 = U_2 < 0 \end{cases} \tag{23}$$

Let us study now the case $n \geq 1$ en detail:
Due to the singular behaviour of the continuous entropy G for $n \geq 2$ we have to distinguish three subcases.
Let us assume first that $1 \leq n < 2$ and $G(u_0)$ is bounded. The latter assumption is satisfied for arbitrary nonnegative $u_0 \in H^{1,2}(\Omega)$. In particular, this also applies to $G(U_h^0)$, i.e. there is a positive constant C_{entrop} such that:

$$\int_\Omega \mathcal{I}_h G(U_h^0) \leq C_{entrop} \qquad \text{independently of } h \to 0$$

Denoting by $R_i^{\mathcal{M}}$ the i-th primitive of $\mathcal{M}^{-1}$, we may write the discrete entropy $G_\sigma(u) := \int_A^u \int_A^r m_\sigma^{-1}(s) ds dr$ as:

$$G_\sigma(u) := \begin{cases} R_2^{\mathcal{M}}(u) - R_2^{\mathcal{M}}(A) - R_1^{\mathcal{M}}(A)(u - A) & \text{if } u \geq \sigma \\ R_2^{\mathcal{M}}(\sigma) - R_2^{\mathcal{M}}(A) - R_1^{\mathcal{M}}(A)(\sigma - A) + \frac{1}{2}(u-\sigma)^2 \mathcal{M}(\sigma)^{-1} + \\ +(u-\sigma)(R_1^{\mathcal{M}}(\sigma) - R_1^{\mathcal{M}}(A)) & \text{if } u < \sigma \end{cases} \tag{24}$$

It is worthwhile to have a closer look on the formula for $G_\sigma(u)$ for $u < \sigma$.
By convexity, $R_2^{\mathcal{M}}(\sigma) - R_2^{\mathcal{M}}(A) - R_1^{\mathcal{M}}(A)(\sigma - A) \geq 0$, and the same property obviously

holds for the second term. The monotonicity of $R_1^{\mathcal{M}}$ implies that the third term is nonnegative, provided $\sigma < A$. The pressure-entropy estimate (18) implies the following estimate:

$$\begin{aligned}\sum_{i=1}^{dim\ V_h} w_i(U^\sigma_{\tau h}(t,x_i)-\sigma)(R_1^{\mathcal{M}}(\sigma)-R_1^{\mathcal{M}}(A)) &\le \int_\Omega \mathcal{I}_h G(U_h^0)\\ &\le C\int_\Omega G_\sigma(U^0(x_i))\\ &\le C_{entrop}\end{aligned} \tag{25}$$

Here, $w_i := \int_\Omega \varphi_i(x)dx$, $i = 1, ...,$dim V_h denotes the mass of a base function $\varphi_i \in V_h$. The regularity of the triangulation implies the existence of positive constants c_1, C_1 such that:

$$c_1 h \le w_i \le C_1 h \qquad i = 1, ..., \text{dim } V_h \tag{26}$$

Since $\mathcal{M}(u) \sim u^n$ in a neighbourhood of zero and $n \ge 1$, we infer for the growth of $R_1^{\mathcal{M}}(\sigma)$ near $\sigma = 0$:

$$R_1^{\mathcal{M}}(\sigma) \sim \begin{cases}\log\sigma & \text{if } n=1\\ -\sigma^{1-n} & \text{if } n>1\end{cases} \tag{27}$$

Using inequalities (25),(26), the following estimate can be established for a negative minimum of $U_{\tau h}(t,.)$:

$$\begin{aligned}C_{entrop} &\ge c_1 h(\min_{x_i}(U^\sigma_{\tau h}(t,x_i))-\sigma)(R_1^{\mathcal{M}}(\sigma)-R_1^{\mathcal{M}}(A))\\ &\ge c_1 h \min_{x_i} U^\sigma_{\tau h}(t,x_i)(R_1^{\mathcal{M}}(\sigma)-R_1^{\mathcal{M}}(A))\end{aligned} \tag{28}$$

which implies:

$$\min(U^\sigma_{\tau h}) \ge \frac{c_1^{-1}C_{entrop}}{h(R_1^{\mathcal{M}}(\sigma)-R_1^{\mathcal{M}}(A))}$$

Letting σ tend to zero, we infer from estimate (27) that the right-hand side in the inequality above tends to zero. Hence for each pair (ε, h) and $1 \le n < 2$ the asserted number $\sigma_0(n,h,\varepsilon,u_0)$ exists.

The subcase $n = 2$ can be handled similarily, provided we guarantee that

$$\limsup_{(\sigma,\tau,h)\to(0,0,0)} \int_\Omega G_\sigma(U^0) < \infty.$$

This can be achieved for instance if u^0 is strictly positive.

Let us study now the subcase $n > 2$ under a strict positivity assumption on u_0. In particular, we have as before:

$$\int_\Omega \mathcal{I}_h G(U_h^0) \le C_{entrop} \qquad \text{independently of } h\to 0$$

Defining for $\beta \in \mathbb{R}$ the set $K_\beta(t) := \{x_i \text{ nodal point: } U^\sigma_{\tau h}(t,x_i) < \beta\}$, we can consequently estimate the cardinality of the set $K_{\alpha\sigma}(t)$ for $0 < \alpha < 1$ as follows:

$$|K_{\alpha\sigma}(t)| \le \frac{c_1^{-1}C_{entrop}}{(1-\alpha)h\sigma(R_1^{\mathcal{M}}(A)-R_1^{\mathcal{M}}(\sigma))}$$

By estimate (27), the term on the right-hand side tends to zero, as $\sigma \to 0$. As a consequence, for σ sufficiently small, $U^\sigma_{\tau h} \geq \alpha\sigma$.

Let us explain now our concept for the case $0 < n < 1$ en detail.
In the sequel, $R_2^{\mathcal{M}}$ denotes the particular second primitive of $\mathcal{M}^{-1}$ characterized by the property:

$$R_2^{\mathcal{M}}(0) = (R_2^{\mathcal{M}})'(0) = 0$$

We define the discrete entropy

$$G_\sigma(u) := \begin{cases} R_2^{\mathcal{M}}(u) & \text{if } u \geq 0 \\ \sigma^{-1} R_2^{\mathcal{M}}(-u) & \text{if } u < 0 \end{cases} \tag{29}$$

and the corresponding discrete mobility $M_\sigma(U)|_E$ as indicated above(see equation (23)). Before proving the nonnegativity result announced in the theorem, let us give – for the reader's convenience – the explicit formula for $M_\sigma(U)|_E$ if $\mathcal{M}(u) = u^n$. For the boundary points x_1, x_2 of E, we abbreviate $U_{\tau h}(t, x_i), i = 1, 2$ by U_1, U_2 as before.

$$M_\sigma(U)\Big|_E := \begin{cases} \frac{(1-n)(U_2-U_1)}{U_2^{1-n}-U_1^{1-n}} & \text{if } \min(U_1, U_2) \geq 0 \\ \frac{(1-n)(U_2-U_1)}{\sigma^{-1}|U_1|^{1-n}+U_2^{1-n}} & \text{if } U_1 < 0 < U_2 \\ \sigma\frac{(n-1)(U_2-U_1)}{|U_1|^{1-n}-|U_2|^{1-n}} & \text{if } \max(U_2, U_1) \leq 0 \\ \mathcal{M}(U_1) & \text{if } U_1 = U_2 \geq 0 \\ \sigma\mathcal{M}(-U_1) & \text{if } U_1 = U_2 < 0 \end{cases} \tag{30}$$

Coming back to the nonnegativity result, we estimate the cardinality of $K_\beta(t)$ for $\beta < 0$ in the following way:

$$|K_\beta(t)| \leq \frac{c_1^{-1} C_{entrop}\sigma}{h R_2^{\mathcal{M}}(\beta)}$$

Arguing as before, the nonnegativity result follows, and for $d = 1$ the theorem is proven.

Eventually, we will consider the case of higher spatial dimensions. Let us indicate the main steps in constructing an admissible entropy-mobility pair (G, M) allowing for nonnegative solutions.
Depending on the mobility's growth exponent n, we start with the same entropy function G_σ as in the one-dimensional case and define functions $f_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ by

$$f_\sigma(a, b) := \begin{cases} (a-b)(G'_\sigma(a) - G'_\sigma(b))^{-1} & \text{if } a \neq b \\ G''_\sigma(a) & \text{if } a = b \end{cases} \tag{31}$$

Let $E \in \mathcal{T}_h$ be given. Let us assume first that there are points $x_0 \in \mathbb{R}^d$ and $(\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ as well as an orthogonal matrix $A \in \mathbb{R}^{d\times d}$ such that $E = x_0 + A\hat{E}_{(\alpha_1,\cdots,\alpha_d)}$ where $\hat{E}_{(\alpha_1,\cdots,\alpha_d)}$ is the convex hull of $(0, \alpha_1 e_1, \cdots, \alpha_d e_d)$.
Then for $U \in V_h$, the restriction of $M(U)$ onto E is given by the formula $M(U)|_E = A\hat{M}(U)|_{\hat{E}} A^{-1}$. Here, $\hat{M}(U)|_{\hat{E}}$ is defined as

$$\hat{M}(U)|_{\hat{E}} := \operatorname{diag}(f_\sigma(U_0, U_1), \cdots, f_\sigma(U_0, U_d))$$

with $U_0 := U(x_0)$ and $U_i := U(x_0 + \alpha_i A e_i)$, $i = 1, \cdots, d$.

Otherwise, we take $g := G'_\sigma$ and define $M(U)|_E$ according to formula (15).

Applying lemma 5.1, the entropy estimate is satisfied by the discrete solutions $(U^\sigma_{\tau h}, P^\sigma_{\tau h})$ corresponding to our construction. Hence, we can repeat the arguments for the nonnegativity results used in the one-dimensional case – with the evident quantitative modifications due to the change in dimension. □

7. Convergence in 1D

In this section, we shall prove the main result on convergence of discrete solutions. Since we will make essential use of the uniform discrete Hölder continuity of approximative solutions – that up to now only could be established in space dimension $d = 1$ – we have to confine ourselves at the moment to the one dimensional case. It is worthwhile to point out that our result has not only its meaning as convergence proof but independently as a new proof for existence of solutions in the continuous setting, too.

We have

Theorem 7.1. *(Convergence result for discrete solutions)*
For $d = 1$, let us assume that $u_0 \in H^{1,2}(\Omega; \mathbb{R}^+_0)$, that the sequences $(\tau_j)_{j\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$, $(\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers monotoneously converge to zero and satisfy for every $j \in \mathbb{N}$: $\tau_j \geq h_j^4$. In addition, we suppose that the corresponding discrete solutions $(U^\sigma_{\tau h}, P^\sigma_{\tau h})$ fulfill the discrete pressure-entropy estimate with uniformly bounded right-hand side, and that σ depending on h is chosen sufficiently small. Then, a subsequence $(U^\sigma_{\tau h}, P^\sigma_{\tau h})$ exists that converges in the following sense to a pair of functions (u, p) which is contained in $C^{1/2,1/8}(\Omega_T; \mathbb{R}^+_0) \cap L^\infty((0,T); H^{1,2}(\Omega)) \cap L^2((0,T); H^2(\Omega)) \times L^2(\Omega_T)$:

- *$U^\sigma_{\tau h} \to u$ uniformly on Ω_T and weakly-$*$ in $L^\infty((0,T); H^{1,2}(\Omega))$.*
- *$P^\sigma_{\tau h} \rightharpoonup p \in L^2(\Omega_T)$ weakly in $L^2((\varepsilon, T); L^2(\Omega))$ for arbitrary $0 < \varepsilon < T$.*
- *$\nabla P^\sigma_{\tau h} \rightharpoonup \nabla p$ weakly in $L^2(S)$ for any $S \subset\subset [u > 0]$ where $[u > 0] := \{(t,x) \in \Omega_T : u(t,x) > 0\}$.*

Furthermore, (u, p) satisfy the entropy estimate

$$\int_{[u(T,.)>0]} G(u(T,.)) + \int_{\Omega_T} p^2 \leq \int_\Omega G(u_0) \tag{32}$$

and solve equation (1) in the following weak sense:

$$\begin{aligned} \int_{\Omega_T} (u - u_0) \frac{\partial}{\partial t} \vartheta \, dx dt &= - \int_{[u>0]} \mathcal{M}(u) \nabla p \nabla \vartheta \, dx dt \\ &\qquad \text{for all } \vartheta \in C^1([0,T]; H^{1,2}(\Omega)) \text{ satisfying } \vartheta(T) = 0 \\ \int_\Omega p(t,x) \psi(x) dx &= - \int_\Omega \nabla u(t,x) \nabla \psi(x) \\ &\qquad \text{for almost all } t \in (0,T) \text{ and every } \psi \in H^{1,2}(\Omega) \end{aligned} \tag{33}$$

Remark: 1. Numerical experiments indicate that the scheme has fine convergence properties also in the case that discrete solutions do not satisfy the pressure-entropy estimate uniformly in $\sigma \to 0$, e.g. if $n \geq 2$ and initial data have compact support. Analytically, we still can prove uniform convergence of discrete solutions, but the limit function does not have $L^2((0,T); H^2(\Omega))$-regularity.

2. In addition, it is possible to prove $u \in H^1((0,T);(H^{1,2}(\Omega))')$ using the methods presented in [1].

Proof. It merely consists of four steps. After having proven the uniform convergence of a subsequence of $(U^\sigma_{\tau h})$ by means of an Arzela-Ascoli argument, we will discuss the limit behaviour of the parabolic part of equation (5). Secondly, we will study the limit behaviour of the discrete mobilities which shall enable us to pass to the limit in the elliptic term. Then, we will prove convergence in equation (6). Finally, the entropy inequality (32) will be established.
As a preliminary observation we remark that the application of theorem 6.1 implies $\lim_{(\tau,h,\sigma)\to 0} \min_{(t,x)\in\Omega_T} U^\sigma_{\tau h}(t,x) \geq 0$ if σ in relation to h is small enough.

Step 1: We consider equation (5), choose different test functions Θ^k and sum over k. Thereby, we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{N-1}\{\left(U^{k+1}-U^k,\Theta^{k+1}\right)_h - \tau(M(U^{k+1})\nabla P^{k+1},\nabla\Theta^{k+1})\} \\ &= \sum_{k=0}^{N-1}(-\left(U^k-U^0,\Theta^{k+1}-\Theta^k\right)_h - \tau(M(U^{k+1})\nabla P^{k+1},\nabla\Theta^{k+1})) + \left(U^N-U^0,\Theta^N\right)_h \\ &= \int_0^{T-\tau} -\left(U^\sigma_{\tau h}-U^0,\partial^+_\tau\Theta_{\tau h}\right)_h - (M(U^\sigma_{\tau h})\nabla P^\sigma_{\tau h},\nabla\Theta_{\tau h})dt \end{aligned} \tag{34}$$

where $\Theta_{\tau h} \in S^{0,-1}(V_h)$ with $\Theta_{\tau h}(T)=0$.
For a function $\vartheta \in C^1((0,T);H^{1,2}(\Omega))$ with $\vartheta(T,.)\equiv 0$, we define $\Theta_{\tau h}|_{I_k} = \mathcal{S}_h\vartheta(t_k,.)$. Here, $\mathcal{S}_h$ stands for the $H^{1,2}(\Omega)$-projection $H^{1,2}(\Omega)\to V_h$.
Let us discuss now the convergence behaviour of $U^\sigma_{\tau h}$. Using lemma 4.2 and corollary 3.2, we observe that the time-interpolations $\tilde{U}^\sigma_{\tau h}$ (for the exact definition, cf. section 1) are uniformly bounded in $C^{1/2,1/8}(\Omega_T)$. Hence, Arzela-Ascoli's theorem guarantees the existence of a subsequence which we still denote by $\tilde{U}^\sigma_{\tau h}$ converging uniformly in $C^{\alpha,\beta}(\Omega_T)$, $\alpha<\frac{1}{2}$, $\beta<\frac{1}{8}$ to a function $u\in C^{1/2,1/8}(\Omega_T)$. Applying lemma 4.2 again, we can estimate $|U^\sigma_{\tau h}(t,x)-\tilde{U}^\sigma_{\tau h}(t,x)| \leq C\tau^{1/8}$ and thus conclude that there exists a subsequence $(U^\sigma_{\tau h})_{(\tau,h,\sigma)\to 0}$ converging uniformly on Ω_T to a function $u\in C^{1/2,1/8}(\Omega_T)$.
Let us consider now convergence in the parabolic term. We already know that

$$|\left(U^\sigma_{\tau h}-U^0,\partial^+_\tau\Theta_{\tau h}\right)_h - (U^\sigma_{\tau h}-U^0,\partial^+_\tau\Theta_{\tau h})| \leq Ch^2\left\|U^\sigma_{\tau h}-U^0\right\|_1\left\|\partial^+_\tau\Theta_{\tau h}\right\|_1.$$

Furthermore, $\partial^+_\tau\Theta_{\tau h}(t,.)$ converges in L^2, uniformly in time, to $\partial_t\vartheta(t,.)$, and therefore we achieve

$$\int_0^{T-\tau}\left(U^\sigma_{\tau h}-U^0,\partial^+_\tau\Theta_{\tau h}\right)_h \to \int_0^T (u-u_0,\partial_t\vartheta)$$

as $\tau,h,\sigma\to 0$.

Step 2: Let us show now that the discrete mobilities $M^\sigma_{\tau h}(U)$ uniformly converge to the original mobility $\mathcal{M}(u)$ as $\tau,h,\sigma\to 0$. For $E_h\in\mathcal{T}_h$, we denote the space-time element $E_h\times[l\tau,(l+1)\tau)$ by $E^l_{\tau h}$, and $M(E^l_{\tau h})$ stands for the discrete mobility on $E^l_{\tau h}$.

We denote the values of $U^\sigma_{\tau h}$ in the end-points of E_h by U_1, U_2 and obtain using the mean-value theorem:

$$M(E^l_{\tau h}) = \frac{1}{G''_\sigma(\xi)} \qquad \text{with } \xi \in [U_1, U_2] \tag{35}$$

If the mobility's growth exponent n satisfies $n \geq 1$, we can easily estimate the difference between $\mathcal{M}(u(t,x)$ and the discrete mobility $M(E^l_{\tau h}(t,x))$ on the the space-time cell, (t,x) is contained in:

$$\begin{aligned} |\mathcal{M}(u(t,x)) - M(E^l_{\tau h}(t,x))| &= |\mathcal{M}(u(t,x)) - m_\sigma(\xi)| \\ &\leq |\mathcal{M}(u(t,x)) - \mathcal{M}(\xi)| + |\mathcal{M}(\xi) - m_\sigma(\xi)| \\ &\leq \sup_{s \in (-\infty, \max U)} |\mathcal{M}'(s)||u(t,x) - \xi| + \mathcal{M}(\sigma) \end{aligned} \tag{36}$$

Due to the uniform discrete Hölder-continuity and the uniform convergence of $U^\sigma_{\tau h}$, for $(\tau, h, \sigma) \to 0$ the terms in the last line of (36) uniformly converge to zero.

If $0 < n < 1$, we argue as follows: For given $\varepsilon > 0$, we find $\delta > 0$ such that $\mathcal{M}(s) < \varepsilon$ for $|s| < 2\delta$. Introducing $S_\delta := \{(t,x) \in \Omega_T : u(t,x) \geq \delta\}$, we obtain repeating the arguments above:

$$|\mathcal{M}(u(t,x)) - M(E^l_{\tau h}(t,x))| \leq \varepsilon \qquad \text{on } S_\delta$$

for τ, h, σ sufficiently small.
The nonnegativity results in the previous chapter imply that we have for sufficiently small τ, σ, h:

$$U^\sigma_{\tau h} \geq -\delta.$$

Hence:

$$|\mathcal{M}(u(t,x)) - M(E^l_{\tau h}(t,x))| \leq \varepsilon + \mathcal{M}(\sigma) \qquad \text{on } \Omega_T \setminus S_\delta.$$

which proves the uniform convergence of the discrete mobilities M.

Step 3: Let us first pass to the limit $\tau, h, \sigma \to 0$ in equation (6). We denote by $\mathcal{R}_h$ the projection $\mathcal{R}_h : L^2((0,T); H^{1,2}(\Omega)) \to S^{0,-1}(V_h)$. From equation (6) we infer that

$$\int_\tau^T (P^\sigma_{\tau h}, \mathcal{R}_h\varphi)_h dt = -\int_\tau^T \int_\Omega \nabla U^\sigma_{\tau h} \nabla \mathcal{R}_h \varphi dx dt \tag{37}$$

for arbitrary $\varphi \in L^2((0,T); H^{1,2}(\Omega))$. On account of inequality (2), the entropy estimate (17), and estimate (3) the term on the left-hand side can be estimated:

$$\begin{aligned} &\left| \int_\tau^T (P^\sigma_{\tau h}, \mathcal{R}_h\varphi)_h dt - \int_\tau^T \int_\Omega P^\sigma_{\tau h} \mathcal{R}_h \varphi dx dt \right| \\ &\qquad \leq Ch ||P^\sigma_{\tau h}||_{L^2((\tau,T);L^2(\Omega))} ||\mathcal{R}_h \varphi||_{L^2((0,T);H^{1,2}(\Omega))} \\ &\qquad \leq Ch ||G(u_0)||_{L^1(\Omega)} ||\varphi||_{L^2((0,T);H^{1,2}(\Omega))} \end{aligned}$$

By use of the following convergence properties of appropriate subsequences:

- $P^\sigma_{\tau h} \rightharpoonup p$ weakly in $L^2((\varepsilon,T); L^2(\Omega))$ for arbitrary $0 < \varepsilon < T$
- $U^\sigma_{\tau h} \rightharpoonup u$ weakly in $L^2((0,T); H^{1,2}(\Omega))$
- $\mathcal{R}_h\varphi \to \varphi$ strongly in $L^2((0,T); H^{1,2}(\Omega))$

we can pass to the limit in equation (37) and obtain for arbitrary $0 < \varepsilon < T$ and arbitrary $\Psi \in L^2((0,T); H^{1,2}(\Omega))$:

$$\int_\varepsilon^T \int_\Omega p(t,x)\Psi(t,x)dxdt = -\int_\varepsilon^T \int_\Omega \nabla u(t,x)\nabla\Psi(t,x)dxdt.$$

This implies:

$$\int_\Omega p(t,x)\psi(x)dx = -\int_\Omega \nabla u(t,x)\nabla\psi(x)dx$$

for almost every $t \in (0,T)$ and arbitrary $\psi \in H^{1,2}(\Omega)$.
From the pressure-entropy estimate we infer additionally that $\|p\|_{L^2((\varepsilon,T);L^2(\Omega))}$ is uniformly bounded for $\varepsilon \to 0$. This implies $p \in L^2(\Omega_T)$ and in particular $p = \Delta u$ in $L^2(\Omega_T)$. Moreover, elliptic regularity theory shows that $u \in L^2((0,T); H^2(\Omega))$.

Let us discuss now the convergence behaviour of the elliptic term in equation (5). Recalling the definition of $\Theta_{\tau h}$, we observe first that $\Theta_{\tau h}$ strongly converges to ϑ in $C^0((0,T); H^{1,2}(\Omega))$. Secondly, the energy estimate (11) implies that $M^{\frac{1}{2}}(U^\sigma_{\tau h})\nabla P^\sigma_{\tau h}$ weakly converges to a function $J \in L^2(\Omega_T)$ with respect to the $L^2(\Omega_T)$-norm.
Let us identify J with $\mathcal{M}^{\frac{1}{2}}(u)\nabla p$. On $\Omega_T \setminus S_\delta$, we may estimate for τ, h, σ sufficiently small:

$$\begin{aligned} &\left|\int_{\Omega_T\setminus S_\delta} M(U^\sigma_{\tau h})\nabla P^\sigma_{\tau h}\nabla\Theta_{\tau h}\right| \\ &\qquad \le ||M^{\frac{1}{2}}(U^\sigma_{\tau h})||_{L^\infty(\Omega_T\setminus S_\delta)}||M^{\frac{1}{2}}(U^\sigma_{\tau h})\nabla P^\sigma_{\tau h}||_{L^2(\Omega_T)}||\Theta_{\tau h}||_{L^2((0,T);H^{1,2}(\Omega))} \\ &\qquad \le C\delta^{n/2}||\Theta_{\tau h}||_{L^2((0,T);H^{1,2}(\Omega))} \end{aligned} \tag{38}$$

This implies that $J \equiv 0$ on $[u = 0]$.

On the other hand, the energy estimate implies that

$$\int_{S_\delta} |\nabla P^\sigma_{\tau h}|^2 dxdt \le C(\frac{2}{\delta})^n$$

on S_δ with a constant C independent of τ, h, σ. Hence, on S_δ $\nabla P^\sigma_{\tau h}$ weakly converges to ∇p with respect to the $L^2(S_\delta)$-norm which proves the assertion.

Step 4: It remains to prove the entropy estimate (32). The following estimates involving the discrete entropy G_σ will be helpful:

i) On $\mathbb{R}^+_0$, we have $G_\sigma(u) \le G(u)$
ii) If $1 \le n < 2$, the discrete entropy G_σ is uniformly Hölder-continuous with exponent $\alpha = 2 - n$ ($\alpha < 1$ if $n = 1$) on the set $(0, \sigma)$.

We have to prove point i) only for $n \ge 1$ and on the set $(0,\sigma)$, since in all the other relevant cases $G_\sigma \equiv G$. Writing down the Taylor-expansion of $R_2^{\mathcal{M}}(u)$

$$R_2^{\mathcal{M}}(u) = R_2^{\mathcal{M}}(\sigma) + (u-\sigma)R_1^{\mathcal{M}}(\sigma) + \frac{1}{2}(u-\sigma)^2\mathcal{M}^{-1}(\sigma) - \frac{1}{6}(u-\sigma)^3(\mathcal{M}(\xi))^{-2}\mathcal{M}'(\xi)$$

with $\xi \in (u,\sigma)$ and using the monotonicity of $\mathcal{M}$, point i) immediately follows.

To prove ii) for $1 < n < 2$, we argue as follows:

$$\begin{aligned}|G_\sigma(u_1) - G_\sigma(u_2)| &= \left|\frac{1}{2}\mathcal{M}^{-1}(\sigma)(u_1-u_2)(u_1+u_2-2\sigma) + (u_1-u_2)R_1^{\mathcal{M}}(\sigma) - (u_1-u_2)R_1^{\mathcal{M}}(A)\right| \\ &\le \sigma\mathcal{M}^{-1}(\sigma)|u_1-u_2|^{2-n}|u_1-u_2|^{n-1} + \\ &\quad + R_1^{\mathcal{M}}(\sigma)|u_1-u_2|^{2-n}|u_1-u_2|^{n-1} + |u_1-u_2|R_1^{\mathcal{M}}(A) \\ &\le C|u_1-u_2|^{2-n}\end{aligned} \tag{39}$$

For the last estimate, we used the growth of $\mathcal{M}^{-1}(\sigma)$ and $R_1^{\mathcal{M}}(\sigma)$ near zero as well as the inclusion $u_i \in (0,\sigma)$, $i = 1,2$. The special case $n = 1$ can be handled similarily.

Let us pass now to the limit in the entropy estimate

$$\int_\Omega \mathcal{I}_h G_\sigma(U_{\tau h}^\sigma(T,.)) + \int_\tau^T (P_{\tau h}^\sigma, P_{\tau h}^\sigma)_h \le \int_\Omega \mathcal{I}_h G_\sigma(U_h^0(.)) \tag{40}$$

First, we show that $\lim_{(\tau,h,\sigma)\to 0}\int_\Omega \mathcal{I}_h G_\sigma(U_h^0(.)) = \int_\Omega G(u_0)$. Since $U_h^0 = \mathcal{I}_h u^0$ is nonnegative (or even strictly positive if $n \ge 2$) and uniformly Hölder continuous on Ω, it will be sufficient to study the convergence behaviour of $G_\sigma(U_h^0)$ for $(\tau,h,\sigma)\to(0,0,0)$. On the set $[u^0 \ge \delta > 0]$, the H^1-convergence of U_h^0 together with the fact that for σ sufficiently small G_σ and G are identical, implies the convergence pointwise almost everywhere for a subsequence of $G_\sigma(U_h^0)$. Using auxiliary estimate i), Lebesgue's theorem of majorized convergence implies the convergence on $[u > 0]$.
We have to discuss the limit behaviour on the set $[u^0 = 0]$ only if $1 \le n < 2$ because otherwise u^0 is either strictly positive or $G_\sigma = G$ on $\mathbb{R}_0^+$. Thus combining auxiliary result ii) with the identity $\lim_{\sigma\to 0} G_\sigma(0) = G(0)$, we observe:

$$|G_\sigma(U_{\tau h}^\sigma(t,x)) - G(0)| \le C|U_{\tau h}^\sigma(t,x)|^\alpha + o_\sigma(1).$$

Hence, for $(\tau,h,\sigma)\to(0,0,0)$ the right-hand side of equation (40) converges to $\int_\Omega G(u^0)$. In the same spirit, we can prove with the help of Fatou's lemma that:

$$\int_{[u(.,T)>0]} G(u(T)) + \int_{\Omega_T} p^2 \le \liminf_{(\tau,h,\sigma)\to(0,0,0)} \int_\Omega \mathcal{I}_h G_\sigma(U_{\tau h}^\sigma(T,.)) + \int_\tau^T (P_{\tau h}^\sigma, P_{\tau h}^\sigma)_h$$

which gives the result. □

8. Convergence results in higher dimensions

In the case of higher dimension, the convergence results to be presented are much weaker than in dimension $d = 1$. This is due to the fact that even in the continuous setting no results in higher space dimensions are known about

- local or global continuity of solutions
- strict positivity on open subsets of the space-time cylinder
- boundedness of solutions

For this reason, we have to confine ourselves to convergence results for the triple $(U_{\tau h}^\sigma, P_{\tau h}^\sigma, J_{\tau h}^\sigma)$ where we denote by $J_{\tau h}^\sigma$ the discrete flux $M(U_{\tau h}^\sigma)\nabla P_{\tau h}^\sigma$. Unfortunately, the identification $\lim_{(\tau,h,\sigma)\to 0} J_{\tau h}^\sigma = \mathcal{M}(u)\nabla p$ remains an open problem.
Our existence result reads:

Theorem 8.1. *Let us assume that*

- $\mathcal{M} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is bounded,*
- $(\tau_k)_{k\in\mathbb{N}}$, $(h_k)_{k\in\mathbb{N}}$, $(\sigma_k)_{k\in\mathbb{N}}$ *are sequences of positive real numbers monotoneously converging to zero,*
- $U_{\tau h}^\sigma$, $P_{\tau h}^\sigma$, $J_{\tau,h}^\sigma$ *are solutions to the discrete system (5), (6),*
- $(U_{\tau h}^\sigma, P_{\tau h}^\sigma)$ *satisfy the discrete pressure-entropy estimate with uniformly bounded right-hand side,*
- σ *depending on h is chosen small enough,*
- $\Omega \subset \mathbb{R}^d$ *is convex and polygonally bounded.*

Then, there is a subsequence $(U_{\tau h}^\sigma, P_{\tau h}^\sigma, J_{\tau,h}^\sigma)_{(\tau,h,\sigma)\to 0}$ *that converges to a triple* $(u, p, J) \in L^2((0,T); H^2(\Omega)) \cap L^\infty((0,T); H^{1,2}(\Omega)) \times L^2(\Omega_T) \times L^2(\Omega_T; \mathbb{R}^d)$ *in the following sense:*

- $U_{\tau h}^\sigma \to u$ *strongly in* $L^2(\Omega_T)$ *and weakly-$*$ in* $L^\infty((0,T); H^{1,2}(\Omega))$.
- $P_{\tau h}^\sigma \rightharpoonup p \in L^2(\Omega)$ *weakly in* $L^2((\varepsilon,T); L^2(\Omega))$ *for arbitrary* $0 < \varepsilon < T$.
- $M(U_{\tau h}^\sigma)\nabla P_{\tau h}^\sigma \rightharpoonup J$ *weakly in* $L^2(\Omega_T; \mathbb{R}^d)$.

Furthermore, u is nonnegative, and equation (1) is satisfied in the following weak sense:

$$\begin{aligned}
&\int_{\Omega_T} (u - u_0)\frac{\partial}{\partial t}\vartheta dx dt = -\int_{[u>0]} J\nabla\vartheta dx dt \\
&\qquad\qquad \textit{for all } \vartheta \in C^1([0,T]; H^{1,2}(\Omega)) \textit{ satisfying } \vartheta(T) = 0 \\
&\int_\Omega p(t,x)\psi(x) dx = -\int_\Omega \nabla u(t,x)\nabla\psi(x) \\
&\qquad\qquad \textit{for almost all } t \in (0,T) \textit{ and every } \psi \in H^{1,2}(\Omega)
\end{aligned} \tag{41}$$

Proof. From lemma 3.3, we infer that

$$\lim_{s\to 0} \int_0^{T-s} (U_{\tau h}^\sigma(t+s,.) - U_{\tau h}^\sigma(t,.))_h dt = 0 \tag{42}$$

uniformly in $(\tau, h, \sigma) \to 0$. Together with the standard estimate $(.,.)_h \le C||.||_{L^2(\Omega)}^2$, we obtain:

$$\lim_{s\to 0} ||U_{\tau h}^\sigma(.+s,.) - U_{\tau h}^\sigma(.,.)||_{L^2((0,T-s);L^2(\Omega))} = 0 \tag{43}$$

uniformly for $(\tau, h, \sigma) \to 0$. Combining this result with the uniform boundedness of $||U_{\tau h}^\sigma||_{L^\infty((0,T);H^{1,2}(\Omega))}$ (cf. inequality (11)) and the following theorem due to J. Simon

Theorem.([19], p.84) *Let* $X \subset B \subset Y$ *with compact imbedding* $X \hookrightarrow B$ *and* $1 \le p \le \infty$. *If* $F \subset L^p(I; X)$ *is bounded and* $\|f(\cdot + h, \cdot) - f(\cdot,\cdot)\|_{L^p(0,T-h,Y)} \to 0$ *uniformly for* $f \in F$ *as* $h \to 0$, *then F is relatively compact in* $L^p(I; B)$.

we observe that a subsequence $(U_{\tau h}^\sigma)$ exists having the convergence behaviour asserted in the theorem.
The entropy estimate implies the existence of a subsequence $(P_{\tau h}^\sigma)$ that weakly converges to a function $p \in L^2(\Omega_T)$ in $L^2((\varepsilon,T); L^2(\Omega))$ for arbitrary $0 < \varepsilon < T$. By the energy estimate, we infer the convergence behaviour conjectured for $J_{\tau h}^\sigma = M(U_{\tau h}^\sigma)\nabla P_{\tau h}^\sigma$. Following the line of argumentation in theorem 7.1, the weak formulation (41) can be established. Having chosen the entropy control parameter σ such that $\liminf_{(\tau,h,\sigma)\to 0} \min(U_{\tau h}^\sigma, 0) = 0$,

the nonnegativity of u follows. Finally, u has $L^2((0,T);H^2(\Omega))$-regularity since the normal derivatives of u vanish, conservation of mass is guaranteed and $p = \Delta u \in L^2(\Omega)$. $\square$

9. Timestep control

One of the most intriguing features in studying fourth order degenerate parabolic equations – with respect to theory as well as to applications such as wetting phenomena – is to trace the solution's free boundary in a correct way.
This section is devoted to a quasi-optimal mechanism of time-step control that allows in each time-step to determine the maximal time increment τ assuring that the numerical free boundary may propagate as fast as its continuous counterpart.
Let us begin with a few remarks about how to determine the discrete solution U.
In each time-step, we have to solve the nonlinear system of equations $B(\bar{U}^{k+1}) = \bar{U}^k$ with $B(\bar{U}) := \left(\mathrm{Id} + \tau_k M_h^{-1}\hat{L}_h(\bar{U})M_h^{-1}L_h\right)\bar{U}$ where $\bar{U}$ is the vector of nodal values corresponding to $U \in V^h$ and $\hat{L}_h(\bar{U}){:=}L_h(U-\alpha)$ (cf. section 2).
In fact, we first consider the corresponding semi-implicite system

$$\left(\mathrm{Id} + \tau_k M_h^{-1}\hat{L}_h(\bar{U}^k)M_h^{-1}L_h\right)\bar{U}^{k+1} = \bar{U}^k$$

and apply an iteration scheme to solve the fully implicite scheme which will be specified later on.
Now observing that in the semi-implicite scheme the numerical free boundary cannot propagate more than a distance h in each time step, it is reasonable to choose the time increment τ smaller than the quotient $\frac{h}{speed(t)}$ where $speed(t)$ stands for the maximal normal velocity of the numerical free boundary at time t. As a consequence of this special choice of time increment, only a very small number of iterations (experiments show < 5) is necessary to obtain the solution of the fully implicite scheme.

Formal considerations – performed in the continuous setting – indicate that the normal velocity $V_n(\xi(t))$ of the free boundary in a point $\xi(t)$ can be related to spatial derivatives of u in $\xi(t)$ according to the following formula:

$$V_n(\xi(t)) = \lim_{x\to\xi(t)} \frac{\mathcal{M}(u(t,x))}{u(t,x)}\frac{\partial}{\partial\nu}\Delta u(t,x) \qquad x \in supp(u(t,.)) \tag{44}$$

For a rigorous proof of this formula in a general setting, we refer to the forthcoming paper([16]). In this section, it will already be verified for self-similar source-type solutions to the equation

$$h_t + \mathrm{div}(h^n \nabla\Delta h) = 0 \tag{45}$$

$$h(t,.) \to \delta_0 \qquad \text{as } t \to 0. \tag{46}$$

Those source-type solutions have been studied by Bernis-Peletier-Williams [8] (space dimension $d = 1$) and Bernis-Ferreira [6] (space-dimension $d > 1$).

But at first, let us make some more remarks related to the discrete setting. In the framework of the algorithm studied in this paper, we formulate a discrete counterpart of formula (44) in the following way:

In a time-step t_k, we first determine on each $E \in \mathcal{T}_h$ numbers

$$v(t_k, E) := \begin{cases} \frac{\mathcal{M}((\overline{U_{\tau h}})_E)}{(\overline{U_{\tau h}})_E}(\sum_{i=1}^d |\partial_{x_i} p|) & \text{if } U_{\tau h}|_E \geq 0 \text{ and } (\overline{U_{\tau h}})_E > 0 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

and then define the time increment by the formula

$$\tau_k := \frac{\gamma h}{0.01 + \max_{E \in \mathcal{T}_h} v(t_k, E)}, \qquad \text{with } \gamma \in (0,1). \tag{48}$$

If $n \geq 1$, the results about Hölder continuity in space for discrete solutions allow to give a robust, but coarse upper bound: $\max_{E \in \mathcal{T}_h} v(t, E) \leq C\|U\|^{n-1}_{L^\infty(\Omega_T)} h^{-5/2}$.
This implies for the time increment:

$$\tau \geq C h^{7/2} \qquad \text{if } n \geq 1 \tag{49}$$

Hence, the assumption $k\tau \geq h^4$ in the previous chapters does not mean a restriction any longer.

Let us prove now the following theorem:

Theorem 9.1. *(Normal velocity of the free boundary for selfsimilar source-type solutions) Let $h : \mathbb{R}^d \times (0, \infty)$ be the solution to equations (45),(46).*
Then the normal velocity of the free-boundary at a free boundary point $\xi(t)$ is given by:

$$V_n(\xi(t)) = \lim_{x \to \xi(t)} h^{n-1} \frac{\partial}{\partial \nu} \Delta h(t, x) \qquad x \in supp(h(t, .)) \tag{50}$$

Proof. Adopting the notation of [6], the solution h of equations (45),(46) can be written as

$$h(t, x) = t^{-d\beta} f(\frac{|x|}{t^\beta}) \qquad \text{with } \beta = \frac{1}{4 + dn} \tag{51}$$

Defining $\eta := \frac{|x|}{t^\beta}$, f solves the equation

$$(\eta^{d-1} f^n (\Delta_\eta f)')' = \beta(\eta^d f)' \qquad \eta > 0 \tag{52}$$

$$\eta^d f(\eta) \to 0 \qquad \text{as} \qquad \eta \to \infty \tag{53}$$

$$\omega_d \int_0^\infty \eta^{d-1} f(\eta) d\eta = 1 \tag{54}$$

where Δ_η is the radial Laplacian and ω_d is the area of the unit sphere in $\mathbb{R}^d$.
In integrated form, equations (52)-(54) read as

$$f^{n-1}(\Delta_\eta f)' = \beta\eta \qquad \text{on } [f > 0] \tag{55}$$

In particular, f is of class C^∞ on $[f > 0]$. By direct computation, we obtain for the normal derivative of Δh on spheres with radius $r = |x|$ around the origin:

$$\frac{\partial}{\partial \nu} \Delta h(t, x) = t^{-\beta(d+3)} \frac{\partial}{\partial r}(\Delta_r f)\bigg|_{rt^{-\beta}} \tag{56}$$

Multiplying this equation by h^{n-1} and using equation (55) yields:

$$\begin{aligned} h^{n-1}(t,x)\frac{\partial}{\partial\nu}\Delta h(t,x) &= t^{-\beta(dn+3)} f(|x|t^{-\beta})^{n-1}(\frac{\partial}{\partial r}\Delta_r f)\Big|_{|x|t^{-\beta}} \\ &= \beta t^{-\beta(dn+4)}|x| \\ &= \beta t^{-1}|x| \end{aligned} \tag{57}$$

For given $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, let us denote by $x(t)$ the intersection of the half-line $T_{x_0} := \left\{x \in \mathbb{R}^d : x = \alpha x_0, \quad \alpha \in \mathbb{R}^+\right\}$ with $S(t)$, the free boundary at time t.
By continuity, we infer:

$$\lim_{x\to x(1)} f^{n-1}(|x|)\frac{\partial}{\partial r}(\Delta_r f)\Big|_{|x|} = \beta|x(1)| \qquad x \in supp(f) \tag{58}$$

which implies for the corresponding moving free boundary point $x(t) := t^\beta x(1)$:

$$\begin{aligned} V_n(x(t)) = \frac{\partial}{\partial t}\langle x(t), \nu_{x(1)}\rangle &= \beta t^{\beta-1}|x(1)| \\ &= \beta t^{-1}|x(t)| \\ &= \lim_{x\to x(t)} h^{n-1}(t,x)\frac{\partial}{\partial \nu}\Delta h(t,x) \qquad x \in supp(h(t,.)) \end{aligned} \tag{59}$$

Here, we used the abbreviation $\nu_x := \frac{x}{|x|}$.
This proves the theorem. □

Remark: Straightforward calculations indicate that the formula for the normal velocity of the free boundary implies that

- the support of solutions does not spread if $n \geq 3$.
- the support does not shrink if $n \geq \frac{3}{2}$,

For further details we refer to [16].

10. Numerical results

The finite element scheme has been implemented in a numerical algorithm and applied to several significant model problems. In each timestep, we have to solve the nonlinear system of equations $B(\bar{U}^{k+1}) = \bar{U}^k$ with

$$B(\bar{U}) \;:=\; \left(\mathrm{Id} + \tau_k M_h^{-1}\hat{L}_h(\bar{U})M_h^{-1}L_h\right)\bar{U}$$

where $\bar{U}$ is the vector of nodal values corresponding to $U \in V^h$ and $\hat{L}_h(\bar{U}){:=}L_h(U-\alpha)$ (cf. section 2). If we consider first the semi-implicit scheme where the mobility is evaluated for fixed $\bar{W}$, we obtain a sparse, linear, nonsymmetric system of equations. For given $\bar{W}$ we look for solutions $\bar{U}$, such that

$$B_l(\bar{W})\bar{U} {:=} \left(\mathrm{Id} + \tau_k M_h^{-1}\hat{L}_h(\bar{W})M_h^{-1}L_h\right)\bar{U} = \bar{U}^k\,.$$

For $d = 1$ the matrix $B_l(\bar{W})$ is a band matrix with bandwidth 5. LR-Decomposition is applied to solve this system of equations with $O(q)$ computational effort, where $q = dim V^h$.
The original nonlinear problem $B(\bar{U}^{k+1}) = 0$ now can be solved either by Newton's method

or by another appropriate fixpoint iteration. Here we calculate for $i \geq 0$ iteratively solutions of

$$B_l(\bar{U}_i^{k+1})\bar{U}_{i+1}^{k+1} = \bar{U}^k$$

where we have defined $\bar{U}_0^{k+1} := \bar{U}^k$. If $\left\|\bar{U}_{i+1}^{k+1} - \bar{U}_i^{k+1}\right\|_\infty$ gets small enough, we select $\bar{U}^{k+1} = \bar{U}_{i+1}^{k+1}$ and continue with the next timestep.
We observe fine convergence properties for this iteration – at most four iterations are necessary to get below a threshold of magnitude 10^{-8}.

Let us describe the results of our numerical experiments. In space dimension $d = 1$, we performed four characteristic simulations, namely

- spreading of self-similar source-type solutions,
- instantaneous development of zero contact angle for initial data with non-zero contact angle and exponent $n < 3$,
- convergence of solutions to a parabolic profile if $n \geq 3$,
- dead core phenomenon (film rupture) for $n = 0.5$ and appropriately chosen initial data.

Let us begin with the detailed description of the first experiment.
Smyth and Hill ([20]) found the following explicit formula for self-similar source-type solutions (cf. figure 2)to equations (45),(46):

$$u(t,x) = \frac{1}{120(t+\tau)^{1/5}}\left[\omega^2 - \frac{x^2}{(t+\tau)^{2/5}}\right]_+^2 \tag{60}$$

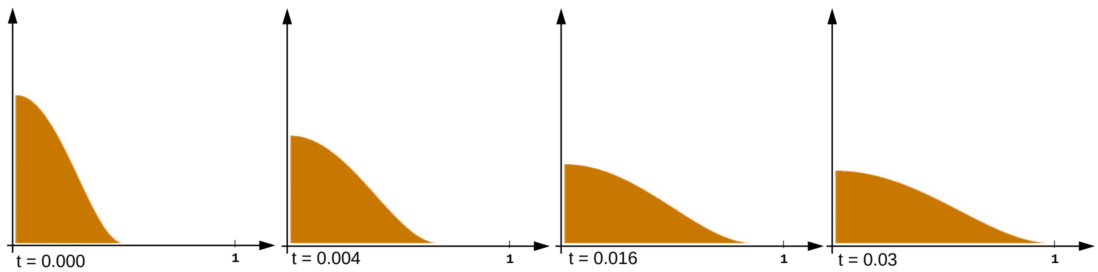

FIGURE 2. Evolution of selfsimilar source-type solution for growth exponent $n = 1$.

Choosing $\omega = 2$, $\tau = 4^{-5}$ and observing the symmetry around zero, it is sufficient to solve equation (1) numerically on $\Omega = (0,1)$ with initial datum:

$$U^0 = \mathcal{I}_h\left(\frac{1}{30}\left[4 - 16x^2\right]_+^2\right). \tag{61}$$

We choose $\sigma = 10^{-8}$ and perform simulations for values of $\gamma \in \{0.1, 0.5, 1.0\}$, where γ is the parameter in the formula for the time-step control (48).
We stop the algorithm at time $T = 0.03$ in order to guarantee that $\mathrm{supp}(u(T,.))$ is contained in $[0,1]$.
The nodal point corresponding to the numerical free boundary is in each time step identified by the formula

$$x_D(t_k) := \inf_{i=1\to\dim \mathrm{V}_h} \{x_i \text{ nodal point } : U(t_k, x_i) \leq 0\}.$$

and compared with the true free-boundary point $x_F(t_k)$. In the following tables, the relevant data for different values of γ and various choices of triangulations are written down.

$\gamma = 1$	1	2	3	4
number of gridpoints	100	200	500	1000
time steps	41	86	261	851
$\Vert\mathcal{I}_h u(0.029,\cdot) - U^N(\cdot)\Vert_{L^\infty(\Omega)}/10^{-3}$	5.47	2.63	0.99	0.47
$\Vert\mathcal{I}_h u - U\Vert_{L^\infty(\Omega_T)}/10^{-3}$	7.61	3.75	1.44	0.67
$\max_{k=1\to N} \vert x_F(t_k) - x_D(t_k)\vert/10^{-3}$	6.23	3.41	1.66	1.25
CPU-time$/s$	2.98	6.16	14.98	44.50

$\gamma = 0.5$	1	2	3	4
number of gridpoints	100	200	500	1000
time steps	85	178	661	1476
$\Vert\mathcal{I}_h u(0.029,\cdot) - U^N(\cdot)\Vert_{L^\infty(\Omega)}/10^{-3}$	4.69	2.31	0.88	0.43
$\Vert\mathcal{I}_h u - U\Vert_{L^\infty(\Omega_T)}/10^{-3}$	6.43	3.17	1.22	0.59
$\max_{k=1\to N} \vert x_F(t_k) - x_D(t_k)\vert/10^{-3}$	6.40	3.32	1.56	1.27
CPU-time$/s$	3.14	6.30	20.28	64.83

$\gamma = 0.1$	1	2	3
number of gridpoints	100	200	500
time steps	539	1104	3939
$\Vert\mathcal{I}_h u(0.029,\cdot) - U^N(\cdot)\Vert_{L^\infty(\Omega)}/10^{-3}$	4.17	2.05	0.80
$\Vert\mathcal{I}_h u - U\Vert_{L^\infty(\Omega_T)}/10^{-3}$	5.49	2.71	1.07
$\max_{k=1\to N} \vert x_F(t_k) - x_D(t_k)\vert/10^{-3}$	8.22	4.09	1.40
CPU-time$/s$	3.54	10.57	77.34

As we do not observe smaller values for U than -7×10^{-6}, the discretization error obviously does not have its maximum at the free boundary, but in the bulk region of the droplet. Furthermore, a comparison of the fourth and fifth line indicates that the maximal error occurs for small values of t. On the contrary, the error in the free boundary is stable over the whole time-interval. In fact, we observe oscillations around zero of the difference $x_F(t) - x_D(t)$ between numerical and exact free boundary as time proceeds – a good indication for the efficiency of our time-step control.
It is remarkable that the time-step control for $\gamma = 1$ allows average values of the time increment τ which are of magnitude h^a with $a \in (1.45, 1.57)$. This is - besides the low numerical cost in each time-step - the main reason for the extremely low CPU-times we

needed for our calculations on a Silicon Graphics Indigo 2 with processor R4000 (250MHz).

The second experiment shows the effect of the time-step control with regard to instantaneous development of zero-contact angle. As initial datum $U^0 : (0,1) \to \mathbb{R}$, we take the nodal projection of the *hat*-shaped function

$$u_0 := \begin{cases} 0.2(x-0.25) & \text{if } 0.25 \le x \le 0.5 \\ 0.2(0.75-x) & \text{if } 0.5 < x \le 0.75 \\ 0 & \text{otherwise.} \end{cases} \tag{62}$$

We choose an uniform triangulation with 100 grid points, take $\gamma = 0.1$, and need a CPU-time of $3.16s$ for the whole sequence presented in figure 3.

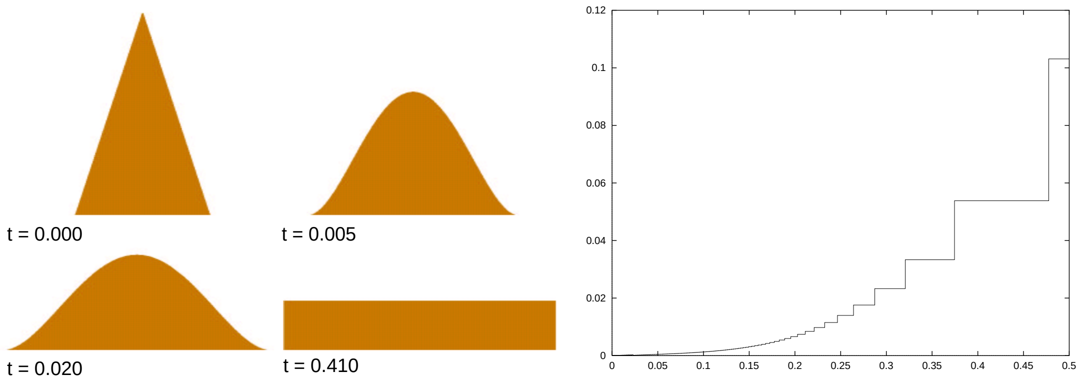

FIGURE 3. Instantaneous development of zero contact angle for initial data with non-zero contact angle, illustrated for $n = 1$ and hat-shaped initial datum with $supp(u_0) = (0.25, 0.75)$ and maximal height 0.05, on the right time–increment versus time depicted in a diagram.

Furthermore, the time increment τ_k is depicted depending on the time-step t_k. As the free boundary initially moves extremely fast, we have to start with very little values τ_k to guarantee the zero-contact angle. Afterwards, the propagation slows down, and larger time increments are sufficient. Finally, when the numerical solution approaches its constant limit value, τ_k reaches its maximum, namely $10h$.

Figure 4 underlines the crucial role of the exponent n in the theory of equation (1). Starting with the same initial data as before, we obtain for $n = 4$ convergence to a solution of Poisson's equation with constant right-hand side as expected by the theoretical results by [2]. Moreover, our experiments show that numerical solutions converge for values of $n \ge 3$ and $t \to \infty$ to a parabolic profile. This gives strong numerical evidence to the conjecture that no spreading of support occurs for values of $n \ge 3$. In addition, we point out, that the algorithm finely works despite the fact that $n \ge 2$ and initial data have compact support.
Theoretical results only assure for values of $n \ge \frac{3}{2}$ that the solution's support cannot shrink as time proceeds. Figure 5 gives numerical evidence that for $n = \frac{1}{2}$ film rupture may occur, a phenomenon that in the literature already has been described by Bertozzi-Pugh ([9]).

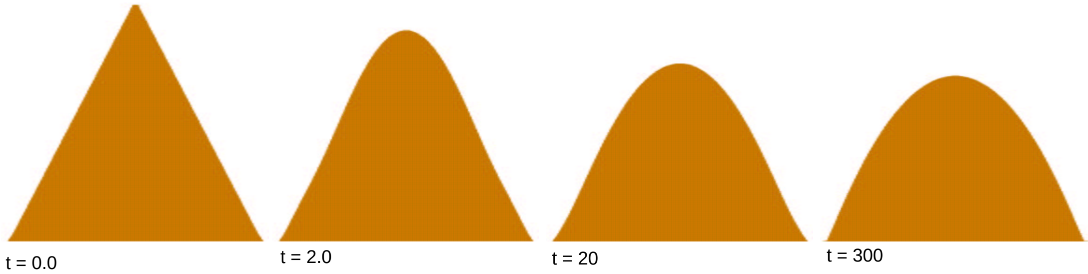

FIGURE 4. No spreading of support, but convergence to a parabolic profile, illustrated for $n = 4$ and same initial datum as in figure 3.

Choosing the initial datum

$$u_0(x) := 10^{-3} + (x - 0.5)^4$$

and $\sigma = 10^{-10}$, we get film rupture for 300 grid points as indicated in figure 5 (CPU-time 25.34s). Finally, figure 6 has been produced using the same initial data as before, but

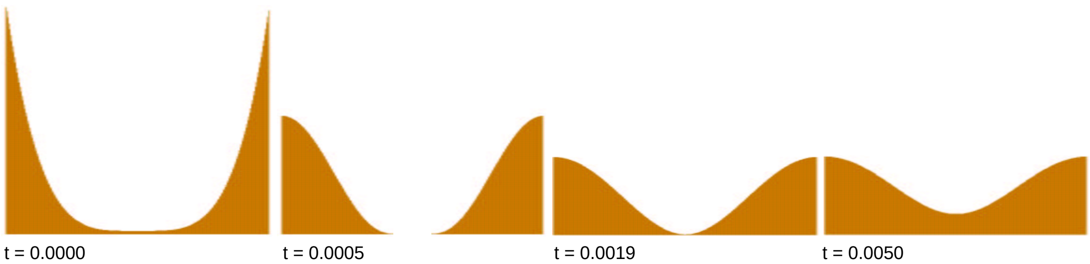

FIGURE 5. Dead core phenomenon, initial datum $u_0(x) = (x-0.5)^4+10^{-3}$, growth exponent $n = 0.5$, using a uniform discretisation in space with 300 gridpoints.

taking $n = 2$ as growth exponent. Obviously, the solution remains strictly positive.

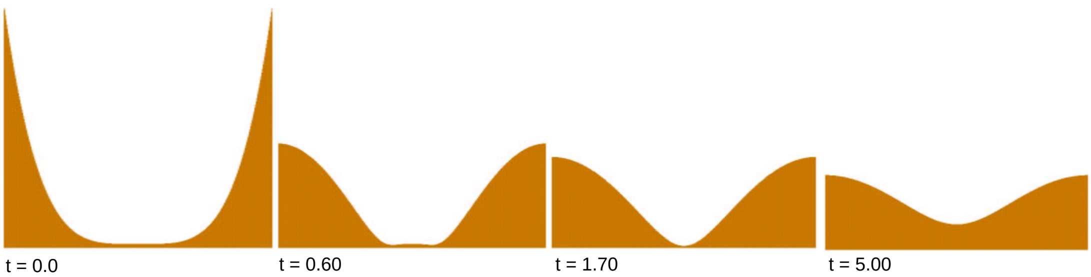

FIGURE 6. No dead core phenomenon, same initial datum as before, but exponent $n = 2$.

Let us discuss now the case of higher spatial dimensions. We subdivide $\Omega = [0, 1]^2$ uniformly using 2500 gridpoints, and perform two characteristic experiments with the semi-implicite algorithm described above – the evolution of *pyramide*-shaped initial data for mobility growth exponents $n = 1$ and $n = 4$ (cf. figures 7 and 8). As the sparsity of $B_l(\bar{W})$ remains, we choose iterative solvers and apply the BiCG-Stab algorithm [22] to solve the corresponding system of equations. It turns out that the number of BiCG-Stab-iterations necessary to get below the error tolerance of 10^{-14} remains low (i.e. in average 70 iterations for the case of 2500 nodal points) as long as the time-step control suggests small time-increments. As soon as the time-increment reaches its maximum of $10h$, we observe in the first experiment an augmentation up to values around 500, whereas in the second experiment the number of iterations remains small. Hence, the CPU-times are rather different – for the first experiment, we need 325s, for the second experiment, 125s are sufficient. Although the numerical cost is still moderate, for further investigations on finer grids and an additional optimization of the algorithm, we intend to develop a multigrid solver as well as an hierarchical preconditioner.

Let us conclude with a remark about the qualitative results of the simulation in higher space dimensions. Figure 7 indicates that the evolution has a regularizing effect on the free boundary when $n < 3$. For larger values of n, the support does not spread as time proceeds – underlining once more the meaning of $n = 3$ as threshold parameter for the qualitative behaviour of solutions to equation (1).

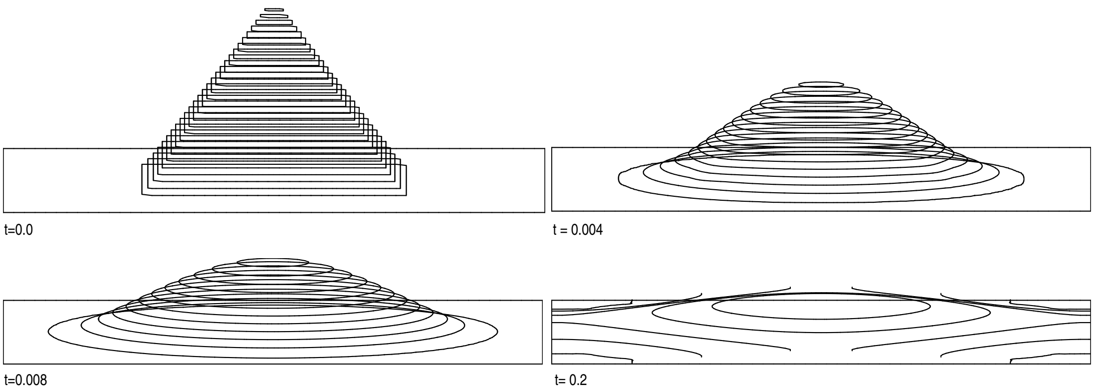

FIGURE 7. 2D analogon to figure 3; please note the evolution's regularizing effect on the numerical free boundary which instantaneously becomes smooth.

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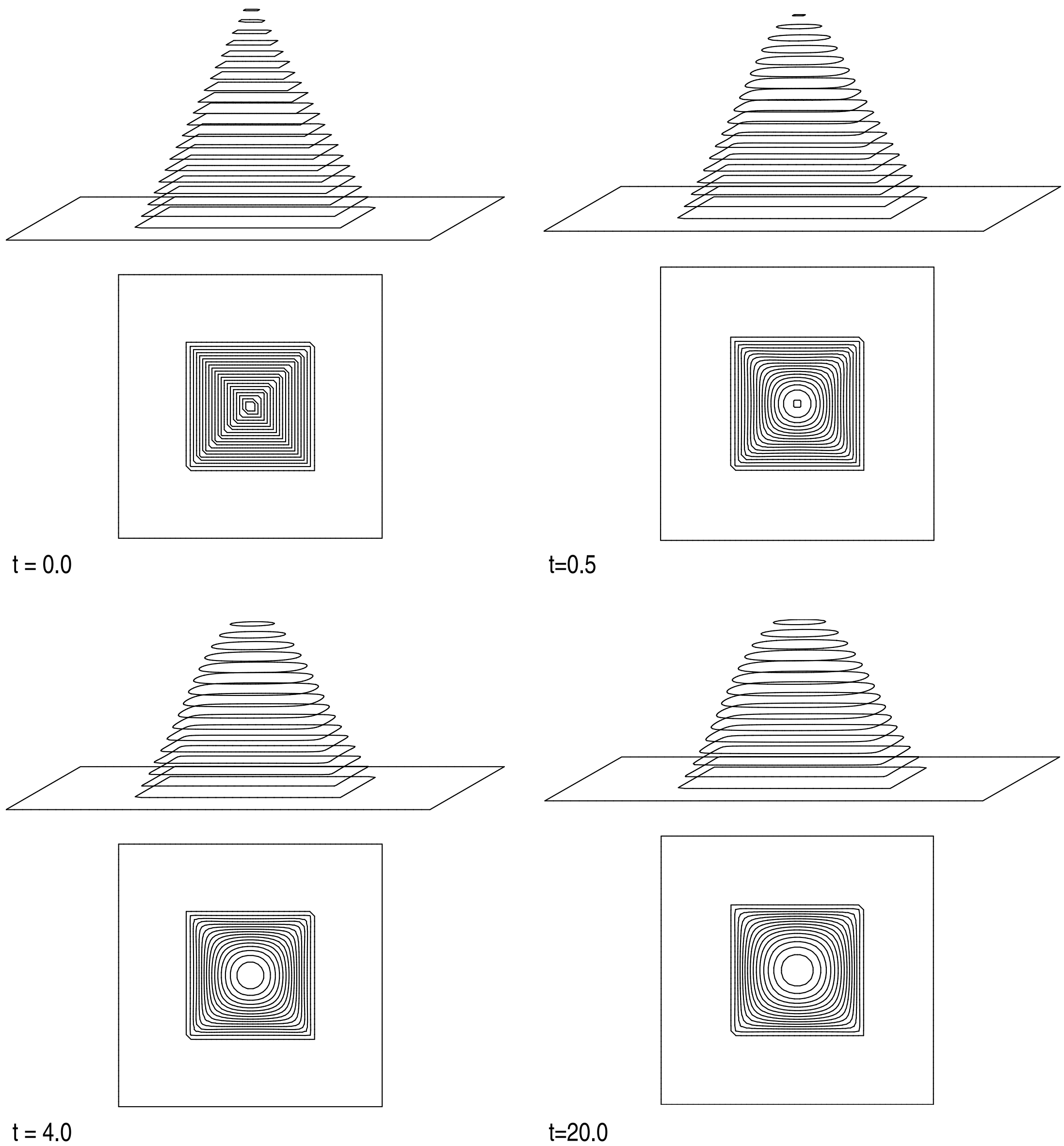

FIGURE 8. 2D analogon to figure 4. The support remains constant in time; its boundary is only Lipschitz continuous.

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