On surfaces of prescribed F-mean curvature

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Abstract

Hypersurfaces of prescribed weighted mean curvature, or F-mean curvature, are introduced as critical immersions of anisotropic surface energies, thus generalizing minimal surfaces and surfaces of prescribed mean curvature. We first prove enclosure theorems in $\mathbb{R}^{n+1}$ for such surfaces in cylindrical boundary configurations. Then we derive a general second variation formula for the anisotropic surface energies generalizing corresponding formulas of do Carmo for minimal surfaces, and Sauvigny for prescribed mean curvature surfaces. Finally we prove that stable surfaces of prescribed F-mean curvature in $\mathbb{R}^3$ can be represented as graphs over a planar strictly convex domain Ω, if the given boundary contour in $\mathbb{R}^3$ is a graph over $\partial\Omega$.

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1 Introduction and main results

Let $X : M \to \mathbb{R}^{n+1}$, $n \geq 2$, be an immersion of class $C^3(M, \mathbb{R}^{n+1})$ of an n-dimensional smooth manifold $M = M^n$ with boundary ∂M into $\mathbb{R}^{n+1}$. We denote the corresponding unit normal by N and the induced area element by dA, and consider general *parametric variational functionals* $\mathcal{F}$ of the form

$$\mathcal{F}(X) := \int_M F(X, N)\, dA. \tag{1.1}$$

The integrand F of class $C^0(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \backslash \{0\}))$ is a *parametric Lagrangian* characterized by the homogeneity condition

$$\text{(H)} \qquad F(y, tz) = tF(y, z) \quad \text{for all} \quad t > 0,\ (y, z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \backslash \{0\}).$$

Note that (H) implies

$$F_{zz}(y,z)z = 0 \quad \text{for all} \quad (y,z) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}), \tag{1.2}$$

hence we will identify the symmetric endomorphism $F_{zz}(y,z) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with its restriction to the space

$$z^{\perp} := \{\zeta \in \mathbb{R}^{n+1} : \langle \zeta, z \rangle = 0\}. \tag{1.3}$$

Important examples of parametric Lagrangians are given by the *area integrand*

$$A(z) := |z|, \tag{1.4}$$

and the integrand

$$E(y,z) := |z| + \langle Q(y), z \rangle \tag{1.5}$$

appearing in the theory of capillary surfaces. Here, Q can be chosen to be a differentiable vectorfield in $\mathbb{R}^{n+1}$ with $\operatorname{div}_{\mathbb{R}^{n+1}} Q(y) = \mathcal{H}(y)$, where $\mathcal{H}(y)$ is a given function representing the prescribed mean curvature. Critical immersions of the corresponding functionals

$$\mathcal{A}(X) := \int_M A(N)\, dA = \int_M dA, \tag{1.6}$$

and

$$\mathcal{E}(X) := \int_M E(X,N)\, dA, \tag{1.7}$$

are *minimal surfaces* and *surfaces of prescribed mean curvature* $\mathcal{H}(X)$, respectively.

Another interesting example is

$$F(z) = \sum_{j=1}^{3} \sqrt{\delta^2 |z|^2 + z_j^2}, \quad \delta > 0, \tag{1.8}$$

which serves as a regularized version of the discrete l^1-norm used for numerical computations involving the anisotropic mean curvature flow [7]. Furthermore, in surface processing [3] such parametric functionals have become an increasingly important tool to enhance edge structures within a suitable surface evolution based on (1.6) and (1.8) . For more examples of integrands and applications in numerical analysis we refer to [8] and [6].

For general parametric integrals we recall the notion of the F*-mean curvature*

$$H_F(X,N) = H_F := -\operatorname{tr}(A_F S), \tag{1.9}$$

as introduced in [2, 4]. Here, $S \in \operatorname{End}(TM)$ is the shape operator defined by $DX \circ S := DN$ on the tangent bundle TM, and $A_F \in \operatorname{End}(TM)$ is the symmetric endomorphism field given by

$$A_F := (DX)^{-1}(F_{zz}(X,N)DX) \quad \text{on} \quad TM. \tag{1.10}$$

For the special parametric Lagrangians in (1.6) and (1.7) the F-mean curvature H_F reduces to the classical mean curvature H, since $A_F|_{T_wM} = \operatorname{Id}_{T_wM}$ for each $w \in M$ and $F(y,z) = A(z)$, or $F(y,z) = E(y,z)$, respectively. Here T_wM denotes the tangent space of M at $w \in M$.

The first author proved in [2] that the Euler equation for $\mathcal{F}$ can be written as

$$H_F = \sum_{i=1}^{n+1} F_{y^i z^i}(X,N). \tag{1.11}$$

Consequently, given a general parametric Lagrangian $F = F(y,z)$, critical immersions of the corresponding parametric functional $\mathcal{F}$ may be viewed as *surfaces of prescribed* F*-mean curvature.* In particular, we will regard critical immersions of the specific parametric functional

$$\mathcal{F}^0(X) := \int_M F(N)\,dA + \int_M \langle Q(X), N\rangle\,dA, \tag{1.12}$$

where $\operatorname{div}_{\mathbb{R}^{n+1}} Q(y) = \mathcal{H}_F(y) \in C^0(\mathbb{R}^{n+1})$ is a given function, as *surfaces of prescribed* F*-mean curvature* $\mathcal{H}_F(X)$. This class of surfaces yields a natural generalization of minimal surfaces if $\mathcal{H}_F(y) \equiv 0$, or of surfaces of constant mean curvature if $\mathcal{H}_F(y) \equiv \mathcal{H}_F^0 \in \mathbb{R}$. Let us point out that the parametric Lagrangian $F(z)$ in (1.12) depends on z only, and that in case $\mathcal{H}_F(y) \equiv \mathcal{H}_F^0 \in \mathbb{R}$ the second integrand in (1.12) is linear in y and z and can be interpreted as a volume term.

As a starting point for our investigations we will derive in Section 2 a differential equation for the surface normal of an arbitrary immersion in terms of the F*-Laplace-Beltrami operator* of X

$$\Delta_F := \operatorname{div}(A_F \operatorname{grad}(.)), \tag{1.13}$$

where the differential operators are taken with respect to the induced metric
(1.14)

$$g(V,W) = g_w(V,W) := \langle DX(V), DX(W)\rangle \quad \text{for} \quad V,W \in T_wM,\ w \in M$$

i.e., $\operatorname{div} = \operatorname{div}_M$ and $\operatorname{grad} = \operatorname{grad}_M$.

Theorem 1.1. *Let N be the normal of an arbitrary immersion X of class $C^3(M,\mathbb{R}^{n+1})$ and let $F \in C^0(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\backslash\{0\}))$ be a parametric Lagrangian. Then*

$$\Delta_F N + \operatorname{tr}(A_F S^2)N = DX(\operatorname{div}(SA_F)). \tag{1.15}$$

Here, $\operatorname{div}(SA_F)$ denotes the divergence of the endomorphism field SA_F, see Section 2 for details.

In Section 3 we consider hypersurfaces with bounded F-mean curvature spanning[1] a given Jordan curve $\Gamma \subset \mathbb{R}^{n+1}$, i.e., we take an immersion $X : M \to \mathbb{R}^{n+1}$ mapping the boundary ∂M topologically onto Γ.

A parametric Lagrangian $F(y,z)$ is said to be *(uniformly) elliptic*, if there exists a constant $M_1 > 0$ such that

$$|z|\langle\zeta, F_{zz}(y,z)\zeta\rangle \geq M_1|\zeta^{\tan}|^2 \tag{E}$$

for all $(y,z) \in \mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\backslash\{0\})$, $\zeta \in \mathbb{R}^{n+1}$, where $\zeta^{\tan} := \zeta - \langle\zeta,z\rangle z/|z|^2$. Notice that $A(z)$ and $E(y,z)$ as defined in (1.4) and (1.5) are elliptic satisfying (E) with $M_1 = 1$.

Surfaces of vanishing F-mean curvature, where F is elliptic, have the *convex hull property* as proven in [2, Thm. 2.3]:

Theorem 1.2. *Let $F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$ be a parametric Lagrangian satisfying* (E). *Suppose $X \in C^0(\overline{M},\mathbb{R}^{n+1}) \cap C^2(M,\mathbb{R}^{n+1})$ is an immersion of vanishing F-mean curvature, i.e., with $H_F(X,N) = \mathcal{H}_F(X) \equiv 0$. If X spans a Jordan curve $\Gamma \subset \mathbb{R}^{n+1}$ contained in the boundary of a closed convex set $K \subset \mathbb{R}^{n+1}$, then $X(\overline{M}) \subset K$.*

For surfaces of bounded (but not necessarily vanishing) F-mean curvature spanning a given Jordan curve Γ within the infinite cylinder

$$\mathcal{Z}_h := \{(x^1,\ldots,x^{n+1}) \in \mathbb{R}^{n+1} : h\sqrt{(x^1)^2+\cdots+(x^n)^2} \leq 1\},\ h \geq 0, \tag{1.16}$$

[1] The existence of conformally parametrized $\mathcal{F}$-minimizing surfaces under Plateau type boundary conditions in was proven in [14] and [15] for $n = 2$ and arbitrary co-dimension, but these solutions might have branch points. For the restricted class of boundary contours considered in Theorem 1.2, White [24] has constructed an embedded $\mathcal{F}$-minimizing disk in $\mathbb{R}^3$.

we restrict our attention to Jordan curves $\Gamma \subset \mathbb{R}^{n+1}$ with an orthogonal projection onto an *h-convex domain* $\overline{\Omega} \subset B_{h^{-1}}(0) \subset \mathbb{R}^n$. Following Sauvigny [21] we call a bounded convex domain $\Omega \subset \mathbb{R}^2$ *κ-convex* for some $\kappa > 0$, if for all $w_0 \in \partial\Omega$ there is a point $\xi_0 = \xi_0(w_0) \in \mathbb{R}^n$ such that the disk $B_{1/\kappa}(\xi_0) \subset \mathbb{R}^n$ contains Ω and such that $w_0 \in \partial B_{1/\kappa}(\xi_0)$.

Theorem 1.3. *Let $F = F(y,z) \in C^0(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\cap C^3(\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus\{0\}))$ be a parametric Lagrangian satisfying* (E). *Suppose $X : \overline{M} \to \mathcal{Z}_h$ of class $C^0(\overline{M},\mathbb{R}^{n+1}) \cap C^2(M,\mathbb{R}^{n+1})$ is an immersion of prescribed F-mean curvature $\mathcal{H}_F \in C^0(\mathbb{R}^{n+1})$, where $\mathcal{H}_F(y)$ satisfies*

$$\|\mathcal{H}_F\|_{C^0(\mathbb{R}^{n+1})} \le M_1 h(n-1), \tag{1.17}$$

and X spans a curve $\Gamma \subset \mathcal{Z}_h$, whose orthogonal projection onto $\mathbb{R}^n$ lies in an h-convex domain $\Omega \subset B_{h^{-1}}(0) \subset \mathbb{R}^n$. Then

$$X(B) \subset \mathcal{Z}_\Omega := \{(x^1,\dots,x^{n+1}) \in \mathbb{R}^{n+1} : (x^1,\dots,x^n) \in \Omega\}. \tag{1.18}$$

In general one cannot expect that surfaces of bounded F-mean curvature satisfying the conditions of Theorem 1.3 can be represented as a graph over the h-convex domain $\Omega \subset \mathbb{R}^n$. For $n = 2$ and *stable* surfaces of bounded mean curvature $\mathcal{H}(y) \in C^{1,\alpha}(\mathbb{R}^3)$, however, Sauvigny was able to prove such a result [21] under a sign condition on $\frac{\partial}{\partial y^3}\mathcal{H}$, and it turns out that the same is true for stable surfaces of prescribed F-mean curvature in $\mathbb{R}^3$, see Theorem 1.4 below.

Before defining stability in Section 4 we generalize do Carmo's [1] second variation formula for the area functional (1.6) to the parametric functional (1.12). That is, we derive a general formula for the second variation $\delta^2\mathcal{F}^0(X,\Xi)$ of the functional (1.12) at critical immersions $X : M \to \mathbb{R}^{n+1}$ in the direction of an arbitrary compactly supported vector field $\Xi \in C^2_0(M,\mathbb{R}^{n+1})$ containing normal and tangential terms[2], see Theorem 4.1. For immersions $X : M \to \mathbb{R}^{n+1}$ of prescribed F-mean curvature $\mathcal{H}_F$, however, the tangential term drops out, see Corollary 4.2, which additionally implies a simplified differential equation for the normal N of such surfaces derived in Corollary 4.3:

$$\Delta_F N + \Big[\mathrm{tr}\,(A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), N\rangle\Big] N = -\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X). \tag{1.19}$$

We generalize Sauvigny's result for surfaces of bounded mean curvature mentioned above to stable surfaces of prescribed F-mean curvature in $\mathbb{R}^3$:

[2]Theorem 4.1 contains as special cases the corresponding second variation formulae of Sauvigny [21] and Räwer [20] who consider only normal, or F-normal variations, respectively

Theorem 1.4. *Let $F = F(z) \in C^0(\mathbb{R}^3) \cap C^3(\mathbb{R}^3 \backslash \{0\})$ be an elliptic parametric Lagrangian satisfying* (E). *Suppose $X : \overline{B} \to \mathcal{Z}_h$ is of class $C^3(B, \mathbb{R}^3) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^3)$ for some $\alpha \in (0,1)$ and a stable immersion of prescribed F-mean curvature $\mathcal{H}_F \in C^{1,\alpha}(\mathbb{R}^3)$, where $\mathcal{H}_F$ satisfies*

$$\|\mathcal{H}_F\|_{C^0(\mathbb{R}^3)} \leq M_1 h, \tag{1.20}$$

and X spans $\Gamma \subset \mathcal{Z}_h$, where Γ is a Jordan curve given as a graph over the boundary $\partial\Omega$ of an h-convex domain $\Omega \subset \mathbb{R}^2$. Then $X(B) \subset \mathcal{Z}_\Omega$, and $X(\overline{B})$ can be represented as a graph over Ω if $\frac{\partial}{\partial y^3}\mathcal{H}_F(y) \geq 0$ for all $y = (y^1, y^2, y^3) \in \mathbb{R}^3$.

The proof of this result can be found in Section 5. For minimal surfaces this result is due to Radó [19]. Gulliver and Spruck [13] generalized Radó's theorem to surfaces of constant mean curvature.

Remark. For simplicity of presentation we have assumed throughout this paper that the surfaces are immersed up to the boundary. The strong smoothness hypotheses of Theorem 1.4, however, allow us to exclude *boundary branch points* for the specific boundary configurations considered in Theorems 1.2 and 1.4 with $n = 2$, see the corresponding remarks in Sections 3 and 5. That is, a *conformally parametrized surface of class $C^{1,\alpha}(\overline{B}, \mathbb{R}^3)$ without interior branch points does not have boundary branch points if it either has vanishing F-mean curvature with boundary contour $\Gamma \subset \partial K$ for some convex set $K \subset \mathbb{R}^3$, or if it has prescribed F-mean curvature $\mathcal{H}_F$ satisfying* (1.20)*, with boundary contour $\Gamma \subset \mathcal{Z}_h$ as in Theorem 1.4.*

A general boundary regularity result, however, guaranteeing $C^{1,\alpha}$-smoothness up to the boundary is currently only available for $\mathcal{F}$-minimizers, see [16], but not for $\mathcal{F}$-critical points.

2 Preliminaries and a differential equation for the normal

In terms of the induced metric $g : T_wM \times T_wM \to \mathbb{R}$ defined in (1.14) we can express an arbitrary tangent vector $V \in T_wM$ as

$$V = g^{kj} g(V, \partial_j)\partial_k, \tag{2.1}$$

and its image under the isomorphism $DX : T_wM \to T_{X(w)}\mathcal{M}$, where $\mathcal{M} := X(\overline{M}) \subset \mathbb{R}^{n+1}$, as

$$DX(V) = g^{kj} g(V, \partial_j)\partial_k X. \tag{2.2}$$

Here g^{kj} are the coefficients of the inverse of the metric tensor g_{kj} and

$$\{\partial_1, \ldots, \partial_n\} := \left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$$

is the coordinate basis spanning $T_w M$. Let $\chi(M)$ be the space of vector fields of class C^2 on M and denote by ∇_V the covariant derivative in the direction of $V \in \chi(M)$, and set $\nabla_i := \nabla_{\partial_i}$, $i = 1, \ldots, n$. We will frequently use the following versions of the product rule

$$U(g(V,W)) = g(\nabla_U V, W) + g(V, \nabla_U W), \tag{2.3}$$

$$\nabla_U(AV) = (\nabla_U A)V + A(\nabla_U V), \tag{2.4}$$

for all $U, V, W \in \chi(M)$ and all differentiable endomorphism fields $A \in \mathrm{End}\,(TM)$. As a consequence of (2.3) we obtain for symmetric $A \in \mathrm{End}\,(TM)$ and $\phi \in C^2(M)$

$$U(d\phi(AV)) = g(\nabla_U(A\,\mathrm{grad}\,\phi), V) + d\phi(A\nabla_U V), \tag{2.5}$$

where $g(\,\mathrm{grad}\,\phi, V) := d\phi(V)$, $V \in T_w M$, defines the gradient of the function ϕ on M as usual. Using the fact that $\langle DX(V), N\rangle = 0$ one can show that

$$U(DX(V)) = DX(\nabla_U V) - \langle DX(V), DX \circ S(U)\rangle N \tag{2.6}$$

for all $U, V \in \chi(M)$. The trace of an endomorphism $A \in \mathrm{End}\,(TM)$ in local coordinates is given by

$$\mathrm{tr}\, A = g^{ik} g(A\partial_i, \partial_k), \tag{2.7}$$

in particular, we will denote

$$\mathrm{tr}\,(A\nabla_\bullet V) = g^{ik} g(A\nabla_i V, \partial_k) \ \text{ for } \ A \in \mathrm{End}\,(TM), V \in \chi(M). \tag{2.8}$$

For $A := \mathrm{Id}$ we obtain the usual *divergence of a vector field* $W \in \chi(M)$

$$\mathrm{div}\, W = \mathrm{div}_M W := \mathrm{tr}\,(\nabla_\bullet W) = g^{ik} g(\nabla_i W, \partial_k). \tag{2.9}$$

The *divergence* $\mathrm{Div}\, Z$ *of a (not necessarily tangential) vector field* $Z : M \to \mathbb{R}^{n+1}$ is given in local coordinates by

$$\mathrm{Div}\, Z = g^{ik}\langle DZ(\partial_i), DX(\partial_k)\rangle, \tag{2.10}$$

and for $Z := DX(W)$, $W \in \chi(M)$ we get $\mathrm{Div}\, Z = \mathrm{div}\, W$ by (2.9) and (2.10). We will also use the notion of the *divergence of an endomorphism field* $\mathrm{div}\, A$, $A \in \mathrm{End}\,(TM)$ with adjoint A^*, given by

$$g(\,\mathrm{div}\, A, V) := \mathrm{tr}\,(\nabla_\bullet A^* V) = g^{ik} g((\nabla_i A^*)V, \partial_k), \tag{2.11}$$

i.e., in local coordinates,

$$\operatorname{div} A = g^{ik}(\nabla_i A)\partial_k, \tag{2.12}$$

where $\nabla_i A$ denotes the covariant derivative of the tensor A, see [9, Def. 2.60].

If we denote the coefficients of the second fundamental form of (M, g) with $h_{ij} := -g(\partial_i, S(\partial_j))$, and, correspondingly, the coefficients of the F-*second fundamental form* by

$$h_{Fij} := -g(\partial_i, A_F S(\partial_j)) = -\langle F_{zz}\partial_i X, \partial_j N\rangle, \tag{2.13}$$

then the F-mean curvature H_F defined in (1.9) can be written as

$$H_F = -\operatorname{tr}(A_F S) = -g^{ij} g(\partial_i, A_F S(\partial_j)) = g^{ij} h_{Fij}. \tag{2.14}$$

Introducing the second order differential operator

$$\Theta_F := \Delta_F - \operatorname{div} A_F, \tag{2.15}$$

where Δ_F is given by (1.13), the first author could prove in [2] that

$$\Theta_F X = H_F N \tag{2.16}$$

holds for any immersion $X \in C^2(M, \mathbb{R}^{n+1})$. This equation reduces to the classical identity $\Delta X = \operatorname{div}\operatorname{grad} X = HN$, if $F(y, z) = A(z)$, or $F(y, z) = E(y, z)$, respectively, see (1.4), (1.5). Moreover, Θ_F is uniformly elliptic if F satisfies the ellipticity condition (E), which leads to the enclosure theorems proven in [2], and which will be used in the proofs of Sections 3 and 5.

Now we will conclude this section with the

Proof of Theorem 1.1: Apply (2.5) to $\phi := N^i$, $i = 1, \dots, n+1$, and $A := A_F \in \operatorname{End}(TM)$ to obtain by (2.6) and (2.4)

$$\begin{aligned} g(\nabla_U(A_F \operatorname{grad} N), V) &\underset{(2.5)}{=} U(DN(A_F V)) - DN(A_F \nabla_U V) \\ &= U(DX(SA_F V)) - DX(SA_F \nabla_U V) \\ &\underset{(2.6)}{=} DX(\nabla_U(SA_F V)) - \langle DX(SA_F V), DX \circ S(U)\rangle N \\ &\quad - DX(SA_F \nabla_U V) \\ &\underset{(2.4)}{=} DX(\nabla_U(SA_F)V) - g(SA_F(V), S(U))N. \end{aligned}$$

Choosing $U = \partial_i$, $V = \partial_k$ we obtain by (1.13), (2.9), (2.12) and (2.7)

$$\begin{aligned}\Delta_F N &\underset{(1.13)}{=} \operatorname{div}(A_F \operatorname{grad} N) \\ &\underset{(2.9)}{=} g^{ik} g(\nabla_i(A_F \operatorname{grad} N), \partial_k) \\ &= g^{ik} DX(\nabla_i(SA_F)\partial_k) - g^{ik} g(SA_F(\partial_k), S(\partial_i))N \\ &\underset{(2.12),(2.7)}{=} DX(\operatorname{div}(SA_F)) - \operatorname{tr}(A_F S^2)N,\end{aligned}$$

where we have used the symmetry of A_F and S to obtain the last term. □

Using the Codazzi equation (cf. [18, p. 30])

$$(\nabla_V S)W = (\nabla_W S)V \tag{2.17}$$

one can show, that

$$\operatorname{div} S = -\operatorname{grad} H. \tag{2.18}$$

Thus in case of the functionals (1.6) or (1.7), where A_F is the identity, we obtain

$$\Delta N + \operatorname{tr}(S^2)N = -DX(\operatorname{grad} H).$$

Therefore (1.15) is a generalization of [21, Hilfssatz 1].

3 Proofs of the enclosure theorems

For the convenience of the reader we recall the proof of Theorem 1.2 from [2].

Proof of Theorem 1.2: Since $H_F(X, N) = \mathcal{H}_F(X) = 0$, we infer from (2.16) that

$$\Theta_F(t(X)) = 0 \tag{3.1}$$

for all affine linear functions

$$t(y) := \langle a, y\rangle + b, \quad a \in \mathbb{R}^{n+1}, b \in \mathbb{R}. \tag{3.2}$$

Taking an arbitrary supporting half plane of the convex body K characterized by an affine linear function t_K, we have $t_K(X) \le 0$ on ∂M, and hence by (3.1) and the maximum principle [11, p. 32], $t_K(X) \le 0$ on $\overline{M}$, i.e.,

$X(\overline{M}) \subset K$. □

REMARK. For $n = 2$, $M := B = B_1(0) \subset \mathbb{R}^2$, and X immersed only in the interior of B but given in conformal parameters, i.e. with

$$|X_u|^2 = |X_v|^2 \quad \text{and} \quad \langle X_u, X_v \rangle = 0 \text{ on } \overline{B}, \tag{3.3}$$

we can exclude boundary branch points. In fact, introducing polar coordinates (r, ϑ) in B and fixing $w_0 \in \partial B$ we can apply Hopf's boundary point lemma [11, p. 34] together with (3.1) to obtain for $t : \mathbb{R}^3 \to \mathbb{R}$ as in (3.2)

$$\frac{\partial}{\partial \nu}[t(X(w))]\big|_{w=w_0} = \langle a, X_r(w_0) \rangle > 0.$$

Since $a \in \mathbb{R}^3$ is arbitrary, we have $|X_r(w_0)| > 0$. Rewriting (3.3) in polar coordinates we conclude $|X_\vartheta(w_0)| > 0$ which shows that w_0 is not a branch point.

PROOF OF THEOREM 1.3: For the function $R(x) := (x^1)^2 + \cdots + (x^n)^2$ we compute similarly as in [5, p.7] using (2.1), (2.15), (2.16), (2.12) and (E)

$$\begin{aligned}
\frac{1}{2}\Theta_F(R(X)) \underset{\substack{(2.1)\\(2.15)\\(2.12)}}{=} & \sum_{i=1}^n X^i \operatorname{div}(A_F \operatorname{grad}(X^i)) \\
& + \sum_{i=1}^n g(\operatorname{grad}(X^i), A_F \operatorname{grad}(X^i)) - \sum_{i=1}^n X^i (\operatorname{div} A_F)(X^i) \\
\underset{(2.15)}{=} & \sum_{i=1}^n X^i \Theta_F(X^i) + \sum_{i=1}^n g(\operatorname{grad}(X^i), A_F \operatorname{grad}(X^i)) \\
\underset{\substack{(2.16)\\(E)}}{\geq} & \sum_{i=1}^n H_F(X, N) X^i N^i + M_1 \sum_{i=1}^n g(\operatorname{grad}(X^i), \operatorname{grad}(X^i)) \\
\geq & -|\mathcal{H}_F(X)|\sqrt{R(X)} + M_1(n-1) \\
\geq & \quad M_1(n-1) - \|\mathcal{H}_F\|_{C^0(\mathbb{R}^3)} h^{-1}
\end{aligned}$$

on M, since $X(\overline{M}) \subset \mathcal{Z}_h$, see (1.16). Notice that we have used the relation $p^{ii} = g(\operatorname{grad}(X^i), \operatorname{grad}(X^i))$, where $P = P(w) = (p^{ij})(w) : \mathbb{R}^{n+1} \to N^\perp(w)$ is the orthogonal projection onto the n-dimensional tangent plane $T_{X(w)}\mathcal{M}$, $\mathcal{M} := X(\overline{M})$, with $\sum_{i=1}^{n+1} p^{ii} = n$, so that

$$\sum_{i=1}^n g(\operatorname{grad}(X^i), \operatorname{grad}(X^i)) = n - g(\operatorname{grad}(X^{n+1}), \operatorname{grad}(X^{n+1})) \geq n - 1.$$

Thus $\Theta_F(R(X)) \geq 0$ on M due to (1.17), and the maximum principle implies $R(X(w)) < h^{-2}$ for all $w \in M$, since $R(X) \not\equiv h^{-2}$ in M. Following Sauvigny [21] we now argue as follows. Assuming that there is some point $w^* \in M$ with $X(w^*) \notin \mathcal{Z}_\Omega$ we infer that $x^* := (X^1(w^*), X^2(w^*)) \notin \Omega$. Let $y^* \in \partial\Omega$ be a point with $|y^* - x^*| = \operatorname{dist}(x^*, \Omega) \geq 0$. (If $x^* \in \partial\Omega$ take $y^* := x^*$.) Since Ω is h-convex there is a point $\eta_* \in \mathbb{R}^n$ such that $\Omega \subset B_{1/h}(\eta_*)$ and $y^* \in \partial B_{1/h}(\eta_*)$. Thus $X(w^*) \notin \mathcal{Z}_{B_{1/h}(\eta_*)}$ and we can look at the 1-parameter family of cylinders $\{\mathcal{Z}(\lambda) := \mathcal{Z}_{B_{1/h}(\lambda\eta_*)}\}_{0\leq\lambda\leq 1}$, for which

$$X(\overline{B}) \subset \mathcal{Z}(0) = \mathcal{Z}_h,$$

and

$$X(B) \cap \partial\mathcal{Z}(1) = X(B) \cap \partial\mathcal{Z}_{B_{1/h}(\eta_*)} \neq \emptyset.$$

By continuity we find $\lambda_0 \in [0,1]$ with

$$X(\overline{B}) \subset \mathcal{Z}(\lambda_0) \text{ and } X(B) \cap \partial\mathcal{Z}(\lambda_0) \neq \emptyset. \tag{3.4}$$

With the same computation as before we deduce for $R_0(x) := (x^1 - \lambda_0\eta_*^1)^2 + \cdots + (x^n - \lambda_0\eta_*^n)^2$ the inequality $\Theta_F(R_0(X)) \geq 0$ on B, hence by (3.4) and the maximum principle $R_0(X(w)) \equiv h^{-2}$, which is absurd. Thus we have shown (1.18). □

4 A general second variation formula and stability

In this section we consider C^3-perturbations $X(.,\epsilon) : M \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^{n+1}$ of an immersed hypersurface $X \in C^3(M, \mathbb{R}^{n+1})$ with

$$X(.,0) = X, \quad \text{and} \tag{4.1}$$

$$\frac{\partial}{\partial\epsilon} X(.,\epsilon)\big|_{\epsilon=0} = \varphi N + DX(V) =: \Xi, \tag{4.2}$$

where $\varphi \in C_0^2(M)$, $V \in \chi(M)$ with compact support. Notice that we admit a non-vanishing tangential component in the variational field $\Xi \in C_0^2(M, \mathbb{R}^{n+1})$ as in [1] but in contrast to [21, p. 64]. The *second variation* $\delta^2\mathcal{F}^0(X,\Xi)$ of the functional $\mathcal{F}^0$ defined in (1.12) at X in the direction of Ξ is defined as

$$\delta^2\mathcal{F}^0(X,\Xi) := \frac{d^2}{d\epsilon^2}\mathcal{F}^0(X(.,\epsilon))\big|_{\epsilon=0}. \tag{4.3}$$

Theorem 4.1. *Let $F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$ be a parametric Lagrangian. Suppose $X \in C^3(M, \mathbb{R}^{n+1})$ is a critical immersion for the functional* (1.12) *and $\Xi \in C_0^2(M, \mathbb{R}^{n+1})$ is a variational field of the form* (4.2). *Then*

$$
\begin{aligned}
\delta^2\mathcal{F}^0(X,\Xi) \;=\; & \int_M \{g(A_F \operatorname{grad}\varphi, \operatorname{grad}\varphi) - \varphi^2(\operatorname{tr}(A_F S^2) \\
& -\langle \nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), N\rangle) + \varphi g(\operatorname{div}(SA_F) + \operatorname{grad}\mathcal{H}_F(X), V)\}\, dA.
\end{aligned} \tag{4.4}
$$

Note that only first order derivatives of $X(.,\epsilon)$ with respect to ϵ, i.e., merely Ξ defined in (4.2) enters the formula for the second variation which justifies the notation on the left-hand side of (4.3).

Proof of Theorem 4.1: Using the identity

$$
\frac{\partial}{\partial\eta}X(.,\epsilon+\eta)_{|_{\eta=0}} = \frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}}
$$

we obtain

$$
\begin{aligned}
\delta^2\mathcal{F}^0(X,\Xi) &= \frac{d^2}{d\epsilon^2}\mathcal{F}^0(X(.,\epsilon))_{|_{\epsilon=0}} \\
&= \frac{d}{d\epsilon}\left(\frac{d}{d\eta}\mathcal{F}^0(X(.,\epsilon+\eta))_{|_{\eta=0}}\right)_{|_{\epsilon=0}} \\
&= \frac{d}{d\epsilon}\left(\delta\mathcal{F}^0(X(.,\epsilon), \frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}})\right)_{|_{\epsilon=0}}.
\end{aligned}
$$

Hence by the first variation formula proven in [2, pp. 5,6] applied to (1.12) and evaluated at $X(.,\epsilon)$ in the direction $\frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}}$,

$$
\begin{aligned}
\delta^2\mathcal{F}^0(X,\Xi) \;=\; & \frac{d}{d\epsilon}\left(\int_M\left\{\left\langle\frac{\partial}{\partial\bar\epsilon}X(.,\bar\epsilon)_{|_{\bar\epsilon=\epsilon}}, N(.,\epsilon)\right\rangle\right.\right. \\
& \left.\left.[\mathcal{H}_F(X(.,\epsilon)) - H_F(X(.,\epsilon), N(.,\epsilon))]\right\} dA\right)_{|_{\epsilon=0}},
\end{aligned} \tag{4.5}
$$

where $N(.,\epsilon)$ is the unit normal and $H_F(X(.,\epsilon), N(.,\epsilon))$ the F-mean curvature of the perturbed immersion $X(.,\epsilon) \in C^3(M, \mathbb{R}^{n+1})$.

According to [2, Lemma 1.1] one has

$$
\frac{\partial}{\partial\epsilon}N(.,\epsilon)_{|_{\epsilon=0}} = -DX(\operatorname{grad}\varphi) + DN(V), \tag{4.6}
$$

where $\varphi \in C_0^2(M)$, $V \in \chi(M)$ with compact support determines the normal and tangential component of Ξ defined in (4.2). From (2.14) on the other hand, we infer

$$\frac{\partial}{\partial\epsilon}H_F(X(.,\epsilon),N(.,\epsilon))|_{\epsilon=0} \underset{(2.14)}{=} \left[\left(\frac{\partial}{\partial\epsilon}g^{ij}(\epsilon)\right)h_{Fij}+g^{ij}\left(\frac{\partial}{\partial\epsilon}h_{Fij}(\epsilon)\right)\right]|_{\epsilon=0}$$

$$=: \quad \mathrm{I}+\mathrm{II}, \tag{4.7}$$

where the argument ϵ indicates that the corresponding quantity belongs to the perturbed immersion $X(.,\epsilon)$. In particular, we write, e.g., $g^{ij}(0)=g^{ij}$, $\partial_l X(.,0)=\partial_l X$, etc. On account of $g^{ij}(\epsilon)g_{js}(\epsilon)=\delta_s^i$ for all $\epsilon\in(-\epsilon_0,\epsilon_0)$ one has

$$\frac{\partial}{\partial\epsilon}g^{ij}(\epsilon) = -g^{ik}(\epsilon)\left(\frac{\partial}{\partial\epsilon}g_{kl}(\epsilon)\right)g^{lj}(\epsilon),$$

and therefore by (4.2) and (2.6) for $U:=\partial_k,\partial_l$, respectively,

$$\begin{aligned}\frac{\partial}{\partial\epsilon}g^{ij}(\epsilon)|_{\epsilon=0} &= -g^{ik}\left\{\left\langle\frac{\partial}{\partial\epsilon}(\partial_k X(.,\epsilon)),\partial_l X\right\rangle+\left\langle\partial_k X,\frac{\partial}{\partial\epsilon}(\partial_l X(.,\epsilon))\right\rangle\right\}|_{\epsilon=0}g^{lj}\\ &\underset{(4.2)}{=} -g^{ik}\left\{\langle\varphi\partial_k N+\partial_k(DX(V)),\partial_l X\rangle+\langle\partial_k X,\varphi\partial_l N+\partial_l(DX(V))\rangle\right\}g^{lj}\\ &\underset{(2.6)}{=} 2\varphi g^{ik}h_{kl}g^{lj}-g^{ik}\langle DX(\nabla_k V),\partial_l X\rangle g^{lj}-g^{ik}\langle DX(\nabla_l V),\partial_k X\rangle g^{lj}.\end{aligned}$$

Thus we obtain for the expression I in (4.7) by the symmetry of the mappings A_F and S

$$\begin{aligned}\mathrm{I} &= \left(\frac{\partial}{\partial\epsilon}g^{ij}(\epsilon)\right)|_{\epsilon=0}h_{Fij}\\ &= 2\varphi g^{ik}h_{kl}g^{lj}h_{Fij}-g^{ik}\langle DX(\nabla_k V),\partial_l X\rangle g^{lj}h_{Fij}\\ &\qquad -g^{ik}\langle DX(\nabla_l V),\partial_k X\rangle g^{lj}h_{Fij}\\ &\underset{(2.13)}{=} 2\varphi g^{ik}g(\partial_k,S(\partial_l))g^{lj}g(\partial_i,A_F S(\partial_j))\\ &\qquad +g^{ik}g(\nabla_k V,\partial_l)g^{lj}g(\partial_i,A_F S(\partial_j))\\ &\qquad +g^{ik}g(\nabla_l V,\partial_k)g^{lj}g(\partial_i,A_F S(\partial_j))\end{aligned}$$

$$
\begin{aligned}
&= 2\varphi g^{ik} g(S(\partial_k), g^{lj} g(SA_F(\partial_i), \partial_j)\partial_l) \\
&\quad + g^{ik} g(SA_F(\partial_i), g^{lj} g(\nabla_k V, \partial_l)\partial_j) \\
&\quad + g^{lj} g(g^{ik} g(\nabla_l V, \partial_k)\partial_i, A_F S(\partial_j)) \\
&\underset{(2.1)}{=} 2\varphi g^{ik} g(A_F S^2(\partial_k), \partial_i) + g^{ik} g(SA_F(\partial_i), \nabla_k V) \\
&\qquad + g^{lj} g(\nabla_l V, A_F S(\partial_j)) \\
&\underset{\substack{(2.7)\\(2.8)}}{=} 2\varphi \,\mathrm{tr}\,(A_F S^2) + \,\mathrm{tr}\,((SA_F + A_F S)\nabla_{\bullet} V).
\end{aligned} \tag{4.8}
$$

Furthermore we need to compute

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon} h_{Fij}(\epsilon)\big|_{\epsilon=0} &\underset{(2.13)}{=} -\left\langle \frac{\partial}{\partial \epsilon}(F_{zz}(N(.,\epsilon))\big|_{\epsilon=0} \partial_i X, \partial_j N \right\rangle \\
&\quad - \left\langle F_{zz}(N)\partial_i \left[\frac{\partial}{\partial \epsilon} X(.,\epsilon)\big|_{\epsilon=0} \right], \partial_j N \right\rangle \\
&\quad - \left\langle F_{zz}(N)\partial_i X, \partial_j \left[\frac{\partial}{\partial \epsilon} N(.,\epsilon)\big|_{\epsilon=0} \right] \right\rangle \\
&\underset{(4.2),(4.6)}{=} -\left\langle \frac{\partial}{\partial \epsilon}(F_{zz}(N(.,\epsilon))\big|_{\epsilon=0} \partial_i X, \partial_j N \right\rangle \\
&\quad - \langle F_{zz}(N)\partial_i \left[\varphi N + DX(V)\right], \partial_j N \rangle \\
&\quad - \langle F_{zz}(N)\partial_i X, \partial_j \left[-DX(\,\mathrm{grad}\,\varphi) + DN(V) \right] \rangle .
\end{aligned} \tag{4.9}
$$

Since $\langle \partial_j N, N \rangle = 0$ we have by (2.6)

$$
\begin{aligned}
\langle F_{zz}(N)\partial_i[\varphi N + DX(V)], \partial_j N \rangle &= \varphi \langle F_{zz}(N)\partial_i N, \partial_j N \rangle \\
&\quad + \langle F_{zz}(N) DX(\nabla_i V), \partial_j N \rangle,
\end{aligned} \tag{4.10}
$$

and also by (2.6) and $DN = DX \circ S$

$$
\begin{aligned}
&\langle F_{zz}(N)\partial_i X, \partial_j[-DX(\,\mathrm{grad}\,\varphi) + DN(V)] \rangle \\
&= -\langle F_{zz}(N)\partial_i X, \partial_j DX(\,\mathrm{grad}\,\varphi) \rangle + \langle F_{zz}(N)\partial_i X, \partial_j DX(S(V)) \rangle \\
&\underset{(2.6)}{=} -\langle F_{zz}(N)\partial_i X, \partial_j DX(\,\mathrm{grad}\,\varphi) \rangle + \langle F_{zz}(N)\partial_i X, DX(\nabla_j(S(V))) \rangle.
\end{aligned} \tag{4.11}
$$

Inserting (4.9)–(4.11) into the expression for II in (4.7) leads to

$$\begin{aligned}
\mathrm{II} \quad = \quad & -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))_{|\epsilon=0}\partial_i X, \partial_j N\right\rangle - \varphi g^{ij}\langle F_{zz}(N)\partial_i N, \partial_j N\rangle \\
& -g^{ij}\langle F_{zz}(N)DX(\nabla_i V), \partial_j N\rangle + g^{ij}\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle \\
& -g^{ij}\langle F_{zz}(N)\partial_i X, DX(\nabla_j(S(V))\rangle \\
\underset{(1.10)}{=} \quad & -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))_{|\epsilon=0}\partial_i X, \partial_j N\right\rangle - \varphi g^{ij} g(A_F S(\partial_i), S(\partial_j)) \\
& -g^{ij} g(A_F(\nabla_i V), S(\partial_j)) + g^{ij}\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle \\
& -g^{ij} g(A_F(\partial_i), \nabla_j(S(V))).
\end{aligned}$$

By the symmetry of A_F and S and by (2.7) (and (2.4) for the last term) we may rewrite this as

$$\begin{aligned}
\mathrm{II} \quad = \quad & -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))_{|\epsilon=0}\partial_i X, \partial_j N\right\rangle - \varphi \operatorname{tr}(A_F S^2) \\
& - \operatorname{tr}(S A_F \nabla_\bullet V) + g^{ij}\langle F_{zz}(N)\partial_i X, \partial_j DX(\operatorname{grad}\varphi)\rangle \\
& - \operatorname{tr}(A_F S \nabla_\bullet V) - \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]).
\end{aligned} \tag{4.12}$$

Adding (4.8) and (4.12) in (4.7) and using the symmetry of F_{zz} we arrive at

$$\begin{aligned}
& \frac{\partial}{\partial\epsilon} H_F(X(.,\epsilon), N(.,\epsilon))_{|\epsilon=0} \\
\underset{(4.8),(4.12)}{=} \quad & \varphi \operatorname{tr}(A_F S^2) + g^{ij}\langle \partial_i X, F_{zz}(N)\partial_j DX(\operatorname{grad}\varphi)\rangle \\
& -g^{ij}\left\langle \frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))_{|\epsilon=0}\partial_i X, \partial_j N\right\rangle - \operatorname{tr}(A_F \circ [(\nabla_\bullet S)V]).
\end{aligned} \tag{4.13}$$

By virtue of (1.10), (2.6), (2.9) and (1.13) we may rewrite the second term on the right-hand side as

$$\begin{aligned}
& g^{ij}\langle \partial_i X, F_{zz}(N)\partial_j DX(\operatorname{grad}\varphi)\rangle \\
= \quad & g^{ij}\langle \partial_i X, \partial_j\{F_{zz}(N)DX(\operatorname{grad}\varphi)\}\rangle \\
& -g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle \\
\underset{\substack{(1.10)\\(2.6)}}{=} \quad & g^{ij} g(\partial_i, \nabla_j(A_F(\operatorname{grad}\varphi))) - g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle \\
\underset{\substack{(2.9)\\(1.13)}}{=} \quad & \Delta_F \varphi - g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle.
\end{aligned} \tag{4.14}$$

Moreover, by the symmetry of F_{zz} we have

(4.15)

$$g^{ij}\langle \partial_i X, \partial_j(F_{zz}(N))DX(\operatorname{grad}\varphi)\rangle = g^{ij}\langle \partial_j(F_{zz}(N))\partial_i X, DX(\operatorname{grad}\varphi)\rangle,$$

and by (2.6), (1.10), (2.4) and (2.11) for general $W \in T_wM$

$$\begin{aligned}
&g^{ij}\langle\partial_i X, \partial_j(F_{zz}(N))DX(W)\rangle \\
&\underset{(2.6)}{=} g^{ij}\langle\partial_j(F_{zz}(N)DX(\partial_i)) - F_{zz}(N)DX(\nabla_j\partial_i), DX(W)\rangle \\
&\underset{(1.10)}{=} g^{ij}\langle\partial_j(DX(A_F(\partial_i))) - DX(A_F(\nabla_j\partial_i)), DX(W)\rangle \\
&\underset{\substack{(2.4)\\(2.6)}}{=} g^{ij}\langle DX((\nabla_j A_F)\partial_i), DX(W)\rangle \\
&= g^{ij}g((\nabla_j A_F)\partial_i, W) \underset{(2.12)}{=} g(\operatorname{div} A_F, W).
\end{aligned} \tag{4.16}$$

Summarizing (4.13), (4.14) and (4.16) for $W := \operatorname{grad}\varphi$ we arrive at

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}H_F(X(.,\epsilon), N(.,\epsilon))\big|_{\epsilon=0} &= \Delta_F\varphi + \varphi\,\mathrm{tr}\,(A_F S^2) - g(\operatorname{div} A_F, \operatorname{grad}\varphi) \\
&\quad - g^{ij}\left\langle\frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon))\big|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle \\
&\quad - \mathrm{tr}\,(A_F \circ [(\nabla_\bullet S)V]).
\end{aligned} \tag{4.17}$$

Now writing out components one calculates

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}(F_{z^l z^k}(N(.,\epsilon)))\big|_{\epsilon=0}\partial_i X^k \partial_j N^l &= F_{z^l z^k z^s}(N)\left(\frac{\partial}{\partial\epsilon}N^s(.,\epsilon)\big|_{\epsilon=0}\right)\partial_i X^k \partial_j N^l \\
&= \partial_j(F_{z^k z^s}(N))\left(\frac{\partial}{\partial\epsilon}N^s(.,\epsilon)\big|_{\epsilon=0}\right)\partial_i X^k,
\end{aligned}$$

whence by (4.6), (4.15) and (4.16) for $W := S(V) - \operatorname{grad}\varphi$,

$$\begin{aligned}
g^{ij}\left\langle\frac{\partial}{\partial\epsilon}(F_{zz}(N(.,\epsilon)))\big|_{\epsilon=0}\partial_i X, \partial_j N\right\rangle &= g^{ij}\langle\partial_j(F_{zz}(N))\partial_i X, DX\circ S(V) \\
&\qquad - DX(\operatorname{grad}\varphi)\rangle \\
&= g(\operatorname{div} A_F, S(V) - \operatorname{grad}\varphi).
\end{aligned} \tag{4.18}$$

Next we claim that for any $V \in T_wM$

$$g(\operatorname{div}(SA_F), V) = \mathrm{tr}\,(A_F \circ [(\nabla_\bullet S)V]) + g(\operatorname{div} A_F, S(V)). \tag{4.19}$$

This together with (4.18) and (4.17) leads to

$$\frac{\partial}{\partial\epsilon}H_F(X(.,\epsilon), N(.,\epsilon))\big|_{\epsilon=0} = \Delta_F\varphi + \varphi\,\mathrm{tr}\,(A_F S^2) - g(\operatorname{div}(SA_F), V). \tag{4.20}$$

By (4.5) we then conclude using (4.2)

$$\begin{aligned}\delta^2\mathcal{F}^0(X,\Xi) &= -\int_M \varphi\{\Delta_F\varphi + \varphi\,\mathrm{tr}\,(A_F S^2) - \varphi\langle\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X),N\rangle \\ &\qquad -g(\,\mathrm{div}\,(SA_F) + \,\mathrm{grad}\,\mathcal{H}_F(X),V)\}\,dA, \end{aligned} \tag{4.21}$$

which proves Theorem 4.1. Notice that the other terms obtained by carrying out the differentiation with respect to ϵ in (4.5) and evaluating at $\epsilon = 0$ vanish, since

$$H_F(X(.,0),N(.,0)) = H_F(X,N) \equiv \mathcal{H}_F(X)$$

because X is a critical immersion for (1.12).

It remains to show (4.19). In fact, by (2.11) and the symmetry of A_F and S

$$\begin{aligned} g(\,\mathrm{div}\,(SA_F),V) &\underset{(2.11)}{=} g^{ik}g(\nabla_i(SA_F)^*V,\partial_k) \\ &= g^{ik}g((\nabla_i A_F^*)S^*V,\partial_k) + g^{ik}g(A_F^*(\nabla_i S^*)V,\partial_k) \\ &\underset{(2.11)}{=} g(\,\mathrm{div}\,A_F,S^*V) + g^{ik}g((\nabla_i S)^*V,A_F(\partial_k)) \\ &= g(S\,\mathrm{div}\,A_F,V) + g^{ik}g(V,(\nabla_i S)A_F(\partial_k)). \end{aligned}$$

The Codazzi equation (2.17) and the symmetry of S and A_F imply now

$$g^{ik}g(V,(\nabla_i S)A_F(\partial_k)) = g^{ik}g(A_F\circ(\nabla_i S)V,\partial_k) \underset{(2.8)}{=} \mathrm{tr}\,(A_F\circ[(\nabla_{\bullet}S)V]),$$

which proves the claim. □

As a consequence of Theorem 4.1 we can state

Corollary 4.2. *Let $X \in C^3(M,\mathbb{R}^{n+1})$ be a immersion of prescribed F-mean curvature $\mathcal{H}_F \in C^1(\mathbb{R}^{n+1})$, where $F = F(z) \in C^0(\mathbb{R}^{n+1})\cap C^3(\mathbb{R}^{n+1}\setminus\{0\})$ is a parametric Lagrangian. Then*

$$\mathrm{div}\,(SA_F) = -\,\mathrm{grad}\,\mathcal{H}_F(X), \quad \text{and} \tag{4.22}$$

$$\begin{aligned}\delta^2\mathcal{F}^0(X,\Xi) &= \int_M \{g(A_F\,\mathrm{grad}\,\varphi,\,\mathrm{grad}\,\varphi) \\ &\qquad -(\,\mathrm{tr}\,(A_F S^2) - \langle\nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X),N\rangle)\varphi^2\}\,dA,\end{aligned} \tag{4.23}$$

where $\Xi = \varphi N + DX(V)$, $\varphi \in C_0^2(M)$, and $V \in \chi(M)$ with compact support. Especially, the second variation of a parametric integrand is depending on normal variations only.

Proof: The symmetry argument we use here is due to White [23]. Consider the surfaces

$$X(\,\cdot\,,\epsilon,\eta) = X + \epsilon(\varphi N + DX(V)) + \eta(\psi N + DX(W))\,,$$

where $\varphi, \psi \in C_0^\infty(M)$ and $V, W \in \chi(M)$ with compact support. Similarly as in (4.5) we have

$$\begin{aligned}
&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\frac{d}{d\eta}\Big|_{\eta=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right] \\
&= \frac{d}{d\epsilon}\left(\int_M \left\{ \left\langle \frac{\partial}{\partial\eta}X(.,\epsilon,\eta)\Big|_{\eta=0}, N(.,\epsilon,0)\right\rangle \right.\right. \\
&\qquad \left.\left. [\mathcal{H}_F(X(.,\epsilon,0)) - H_F(X(.,\epsilon,0), N(.,\epsilon,0))]\right\} dA\right)\Big|_{\epsilon=0}. \qquad (4.24)
\end{aligned}$$

Hence by (4.20) we obtain

$$\begin{aligned}
&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\frac{d}{d\eta}\Big|_{\eta=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right] \\
&= \int_M \psi\{-\Delta_F\varphi - \varphi\,\mathrm{tr}\,(A_F S^2) + g(\,\mathrm{div}\,(SA_F), V)\} \\
&\qquad + \psi\langle \nabla_{\mathbb{R}^{n+1}}\mathcal{H}_F(X), \varphi N + DX(V)\rangle\, dA.
\end{aligned}$$

Since

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\frac{d}{d\eta}\Big|_{\eta=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right] = \frac{d}{d\eta}\Big|_{\eta=0}\left[\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{F}^0(X(.,\epsilon,\eta))\right],$$

we arrive at

$$\begin{aligned}
&\int_M \psi g(\,\mathrm{div}\,(SA_F), V) + \psi g(V, \,\mathrm{grad}\,\mathcal{H}_F(X))\, dA \\
&= \int_M \varphi g(\,\mathrm{div}\,(SA_F), W) + \varphi g(W, \,\mathrm{grad}\,\mathcal{H}_F(X))\, dA \qquad (4.25)
\end{aligned}$$

for all $\varphi, \psi \in C_0^\infty(M)$ and $V, W \in \chi(M)$ with compact support, where we used that

$$\int_M \psi(-\Delta_F\varphi - \varphi\,\mathrm{tr}\,(A_F S^2))\, dA = \int_M \varphi(-\Delta_F\psi - \psi\,\mathrm{tr}\,(A_F S^2))\, dA.$$

This is only possible if $\mathrm{div}\,(SA_F) = -\,\mathrm{grad}\,\mathcal{H}_F(X)$, for if not, we could choose $W \equiv 0$ to have a vanishing right-hand side in (4.25), and ψ and V appropriately to obtain a positive left-hand side and thus a contradiction. □

Inserting (4.22) into the formula (1.15) of Theorem 1.1 for the normal of an $\mathcal{F}^0$-critical immersion we obtain

COROLLARY 4.3. *Let N be the normal of an immersion X of class $C^3(M,\mathbb{R}^{n+1})$ of prescribed F-mean curvature $\mathcal{H}_F(y) \in C^1(\mathbb{R}^{n+1})$, where $F = F(z)$ is a parametric Lagrangian of class $C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$. Then*

$$\Delta_F N + (\operatorname{tr}(A_F S^2) - \langle \nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X), N\rangle) N = -\nabla_{\mathbb{R}^{n+1}} \mathcal{H}_F(X). \tag{4.26}$$

The above corollary generalizes [21, Satz 1].

The notion of stability is defined as follows.

DEFINITION 4.4. *Let $X \in C^3(M,\mathbb{R}^{n+1})$ be an $\mathcal{F}^0$-critical immersion, where $\mathcal{F}^0$ is defined in* (1.12) *with a parametric Lagrangian $F = F(z) \in C^0(\mathbb{R}^{n+1}) \cap C^3(\mathbb{R}^{n+1}\backslash\{0\})$. Then X is called* strictly stable *if $\delta^2\mathcal{F}^0(X,\Xi) > 0$ for all $\Xi \in C^2_0(M,\mathbb{R}^{n+1})$. If $\delta^2\mathcal{F}^0(X,\Xi) \geq 0$ we say X is* stable.

5 Graph representation of prescribed F-mean curvature surfaces

The proof of Theorem 1.4 is based on a maximum principle for elliptic equations of the form $Lu = (\alpha^{ij}u_{x^j})_{x^i} + \beta^i u_{x^i} + cu$. Usually it is required that the coefficient c be non-positive. As was pointed out in [12] this condition may be replaced by assuming that the first eigenvalue of L is positive. The following lemma is related to [12, Lemma 1], but we weaken this assumption to a merely non-negative eigenvalue and assume less regularity of the coefficients:

LEMMA 5.1. *Let $Lu = (\alpha^{ij}u_{x^j})_{x^i} + \beta^i u_{x^i} + cu \leq 0$ be a linear elliptic equation in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where α^{ij}, β^i, $c \in C^{0,1}(\overline{\Omega})$, $i,j = 1,\ldots,n$. Assume that the first eigenvalue λ of L is non-negative on Ω. If for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ we have $Lu \leq 0$ and $u_{|\partial\Omega} > 0$, then* $\inf_\Omega u > 0$.

PROOF: On account of the continuity of u we can assume that there is a smoothly bounded domain $\Omega_2 \subset\subset \Omega$ with $u_{|\partial\Omega_2} > 0$. The first eigenvalue λ of L in a domain Ω_1 with $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$ is simple and therefore strictly positive, since otherwise we could apply [11, Thm. 8.38] to find a positive eigenfunction $\xi \in \overset{\circ}{H}{}^{1,2}(\Omega_1)$ on Ω_1 with $\xi = 0$ on $\partial\Omega_1$. Extending ξ by zero outside of Ω_1 we would obtain $\tilde{\xi} \in \overset{\circ}{H}{}^{1,2}(\Omega)$ with $L\tilde{\xi} + \lambda\tilde{\xi} = 0$, $\tilde{\xi} \not\equiv 0$ on Ω with $\tilde{\xi} \equiv 0$ on $\Omega\backslash\Omega_1$, contradicting [11, Thm. 8.38].

The regularity of the coefficients α^{ij}, β^i, c, leads to $\xi \in C^{1,\alpha}(\overline{\Omega}_1)$, see e.g. [10, Ch. III]. Thus in Ω_2 we can write $u = \xi\, v$, and due to the regularity of

u and ξ we obtain a.e. on Ω_2:

$$\begin{aligned} L(\xi v) &= v(\alpha^{ij}\xi_{x^j})_{x^i} + \xi(\alpha^{ij}v_{x^j})_{x^i} + 2\alpha^{ij}v_{x^i}\xi_{x^j} + v\beta^i\xi_{x^i} + \xi\beta^i v_{x^i} + cv\xi \\ &= vL\xi + \xi((\alpha^{ij}v_{x^j})_{x^i} + \beta^i v_{x^i}) + 2\alpha^{ij}v_{x^i}\xi_{x^j} \\ &= \xi(\ (\alpha^{ij}v_{x^j})_{x^i} + \beta^i v_{x^i} + (2/\xi)\ \alpha^{ij}\xi_{x^j}\ v_{x^i} - (\lambda/\xi)\, v)\,. \end{aligned}$$

Thus we obtain an elliptic differential inequality for v:

$$0 \le \int_{\Omega_2} [\alpha^{ij}v_{x^j}\varphi_{x^i} - (\tilde{\beta}^i v_{x^i} - (\lambda/\xi)\, v)\varphi]\, dx$$

for all non-negative $\varphi \in C_0^1(\Omega_2)$, where $\tilde{\beta}^i := \beta^i + (2/\xi)a^{ij}\xi_{x^j}$. (Note that $\tilde{\beta}^i$ and $\lambda/\xi \in L^\infty(\Omega_2)$.) Thus the weak maximum principle [11, Thm 8.1] holds for v and we have $\inf_{\Omega_2} v \ge 0$. By the strong minimum principle [11, Thm. 8.19] we obtain the strict relation $\inf_{\Omega_2} v > 0$. □

Proof of Theorem 1.4: According to our assumptions on Γ and Ω in Theorem 1.3 there is a function $f \in C^2(\partial\Omega)$, such that $\Gamma = \{(x, f(x)) : x \in \partial\Omega\}$ is (positively) oriented by setting

$$P_k := (x_k, f(x_k)), \quad k = 1, 2, 3, \tag{5.1}$$

where $x_k \in \partial\Omega$, $k = 1, 2, 3$, are chosen in positive orientation with respect to $\mathbb{R}^2$.

Since X is immersed on $\overline{B}$ we may assume without loss of generality that X is conformally parametrized, i.e. satisfies the conformality relations (3.3). (Otherwise we can perform a diffeomorphism $w : \overline{B} \to \overline{B}$ of class $C^{2,\alpha}(B, \mathbb{R}^2) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^2)$ such that $\tilde{X} := X \circ w^{-1}$ is conformally parametrized, see e.g. [17, Corollary 3.1.2].) Performing a suitable Möbius transformation on $\overline{B}$ we may assume that X satisfies the three-point condition

$$X(w_k) = P_k \quad k = 1, 2, 3, \tag{5.2}$$

where w_1, w_2, w_3 are fixed distinct points on ∂B.

Fix some point $w_0 = e^{i\vartheta_0} \in \partial B$. Since Ω is h-convex there is a point $\eta_0 \in \mathbb{R}^2$ such that $\Omega \subset B_{1/h}(\eta_0)$ and such that $y_0 := (X^1(w_0), X^2(w_0)) \in \partial\Omega$ is contained in $\partial B_{1/h}(\eta_0)$. Without loss of generality we may assume that $\eta_0 = 0$. By Hopf's boundary point lemma we then obtain for the function $R(x) := (x^1)^2 + (x^2)^2$

$$\frac{\partial}{\partial\nu} R(X(w))|_{w=w_0} > 0,$$

i.e., in polar coordinates (r, ϑ)

$$2\sum_{i=1}^{2} X^i(w)\frac{\partial}{\partial r}X^i(w)_{|_{w=w_0}} > 0, \tag{5.3}$$

which implies $|X_r(w_0)| > 0$, and by conformality also

$$|X_\vartheta(w_0)| > 0. \tag{5.4}$$

Since $R(X(w_0)) = h^{-2} \geq R(X(w))$ for all $w = e^{i\vartheta} \in \partial B$ we obtain

$$0 = \frac{\partial}{\partial\vartheta}_{|\vartheta=\vartheta_0}(R(X(e^{i\vartheta}))) = \sum_{i=1}^{2} X^i(e^{i\vartheta})\frac{\partial}{\partial\vartheta}X^i(e^{i\vartheta})_{|_{\vartheta=\vartheta_0}}. \tag{5.5}$$

Since $\Gamma = \{(x, f(x)) : x \in \partial\Omega\}$, $f \in C^2(\partial\Omega)$, we have

$$|X^3_\vartheta| = |f_{x^1}X^1_\vartheta + f_{x^2}X^2_\vartheta| \leq \|\nabla\tilde{f}\|_{C^0(\mathbb{R}^2)}\sqrt{(X^1_\vartheta)^2 + (X^2_\vartheta)^2},$$

where $\tilde{f} \in C^2(\mathbb{R}^2)$ is an extension of f onto $\mathbb{R}^2$ with controlled C^2-norm, see [11, p. 137]. Hence, by (5.4)

$$0 < |X_\vartheta(w)|^2_{|_{w=w_0}} \leq (1 + \|\nabla\tilde{f}\|^2_{C^0(\mathbb{R}^2)})\left[(X^1_\vartheta(w))^2 + (X^2_\vartheta(w))^2\right]_{|_{w=w_0}}. \tag{5.6}$$

By (5.2) the mapping $(X^1, X^2) : \partial B \to \partial\Omega$ respects the positive orientation, thus we infer from (5.5) and (5.6) that there is a constant $\sigma > 0$ such that

$$X^1_\vartheta(w_0) = -\sigma X^2(w_0), \quad X^2_\vartheta(w_0) = \sigma X^1(w_0).$$

Therefore, by (5.3)

$$X^1_r(w_0)X^2_\vartheta(w_0) - X^2_r(w_0)X^1_\vartheta(w_0) = \sigma(X^1(w_0)X^1_r(w_0) + X^2(w_0)X^2_r(w_0)) \underset{(5.3)}{>} 0,$$

which means that $X^1_u(w_0)X^2_v(w_0) - X^1_v(w_0)X^2_u(w_0) > 0$, i.e., the third component of N^3 of the normal N is positive on ∂B. Moreover, by Corollary 4.3, N^3 satisfies the elliptic differential equation

$$\Delta_F N^3 + (\,\mathrm{tr}\,(A_F S^2) - \langle\nabla_{\mathbb{R}^3}\mathcal{H}_F(X), N\rangle)N^3 = -\frac{\partial\mathcal{H}_F}{\partial y^3}(X).$$

Using the assumption on $\mathcal{H}_F(X)$ this relation is given in coordinates by

$$\mathcal{L}N^3 := \partial_i(\sqrt{g}g^{ij}a_{jk}g^{kl}\partial_l N^3) + \sqrt{g}(\,\mathrm{tr}\,(A_F S^2) - \langle\nabla_{\mathbb{R}^3}\mathcal{H}_F(X), N\rangle)N^3 \leq 0\,,$$

which we regard as a linear elliptic equation for N^3 with the differential operator $\mathcal{L}$ associated to the second variation formula (4.23). Here, $g = \det(g_{ij})$, $a_{jk} = \langle F_{zz}(N)\partial_j X, \partial_k X\rangle$ and the remaining coefficients are of class $C^{0,\alpha}(\overline{B})$. Since X is stable we have $\delta^2\mathcal{F}^0(X,\Xi) \geq 0$, hence the first eigenvalue of $\mathcal{L}$ is non-negative. Thus Lemma 5.1 is applicable and we have $N^3 > 0$ on B.

Since $X : \partial B \to \Gamma$ is a topological mapping, we can apply Sauvigny's reasoning involving degree theory as in [21, pp. 53,54] to conclude the proof. □

Remark. We have seen in (5.3) and (5.4) that there are no branch points on the boundary by the simple Hopf maximum principle argument, which is applicable because of our regularity assumptions up to the boundary. Consequently, it would suffice to assume that X is conformal and has no *interior* branch points and maps the boundary ∂B only weakly monotonically onto Γ, but at this point it is an open question if one can relax the smoothness assumptions to $X \in C^0(\overline{B},\mathbb{R}^3) \cap C^3(B,\mathbb{R}^3)$.

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