Abstract. In this paper the space of images is considered as a Riemannian manifold using the metamorphosis approach [24, 34, 35], where the underlying Riemannian metric simultaneously measures the cost of image transport and intensity variation. A robust and effective variational time discretization of geodesics paths is proposed. This requires to minimize a discrete path energy consisting of a sum of consecutive image matching functionals over a set of image intensity maps and pairwise matching deformations. Under low regularity requirements for the input images, the existence of discrete geodesic paths defined as minimizers of this variational problem is shown. Furthermore, Γ-convergence of the underlying discrete path energy to the continuous path energy is proved. This includes a diffeomorphism property for the induced transport and the existence of a square-integrable weak material derivative in space and time. A spatial discretization via finite elements combined with an alternating descent scheme in the set of image intensity maps and the set of matching deformations is presented to approximate discrete geodesic paths numerically. Computational results underline the efficiency of the proposed approach and demonstrate important qualitative properties.

1. Introduction. The study of spaces of shapes from the perspective of a Riemannian manifold allows to transfer many important concepts from classical geometry to these usually infinite-dimensional spaces. During the past decade, this Riemannian approach had an increasing impact on the development of new methods in computer vision and imaging, ranging from shape morphing and modeling, e.g. [18], and shape statistics, e.g. [14], to computational anatomy [4]. A variety of Riemannian shape spaces has been investigated in the literature. Some of them are finite-dimensional and consider polygonal curves or triangulated surfaces as shapes [18, 20], but most approaches deal with infinite-dimensional spaces of shapes. Prominent examples with a fully flexed geometric theory are spaces of planar curves with curvature-based metric [21], elastic metric [31], or Sobolev-type metric [10, 22, 33]. The concept of optimal transport was used to study the space of images, where image intensity functions are considered as probability measures, e.g. Zhang et al. [38] minimize the Monge-Kantorovich functional \( \int_{D} |\psi(x) - x|^2 \rho_0(x) \, dx \) over all mass preserving mappings \( \psi : D \to D \). Benamou and Brenier [6] used a flow reformulation of optimal transport, which nicely fits into the Riemannian context.

For only a few nontrivial application-oriented Riemannian spaces geodesic paths can be computed in closed form (e.g. [37, 32]), else the system of geodesic ODEs has to be solved using numerical time stepping schemes (e.g. [19, 5]). Alternatively, geodesic paths connecting shapes can also be approximated via the minimization of discretized path length [30] or path energy functionals [15, 36]. In this paper, we will develop such a variational time discretization on the space of images using the metamorphosis concept proposed by Trouvé and Younes [35, 34, 16]. This concept is a generalization of the flow of diffeomorphism approach initiated by Dupuis, Grenander and Miller [13]. Thus, in what follows, we will briefly review these two approaches as a basis for the discussion of our time discrete metamorphosis model and the Γ-convergence analysis to be presented in this paper.

Flow of diffeomorphism. Here, we give a very short exposition and refer to [13, 5, 17, 23] for more details. Following the classical paradigm by Arnold [1, 2], one studies the temporal change of image intensities from the perspective of a family of diffeomorphisms \( (\psi(t))_{t \in [0,1]} : \overline{D} \to \mathbb{R}^d \) on the closure of the image domain \( D \subset \mathbb{R}^d \) for \( d = 2,3 \) describing a flow, which transports image intensities along particle paths. In what follows, we suppose that \( D \) is a bounded domain with Lipschitz boundary. A path energy

\[
E[(\psi(t))_{t \in [0,1]}] = \int_0^1 \int_D L[v(t), v(t)] \, dx \, dt
\]
is associated which each path \( (\psi(t))_{t \in [0,1]} \) in the space of images, where \( v(t) = \psi(t) \circ \psi^{-1}(t) \) represents the Eulerian velocity of the underlying flow and \( L \) is a quadratic form corresponding to a higher order elliptic operator. Physically, the metric \( g_{\psi(t)}(\psi(t), \dot{\psi}(t)) = \int_D L[v(t), v(t)] \, dx \) describes the viscous dissipation in a multipolar fluid model as investigated by Nečas and Šilhavý [26]. From this perspective, a suitable choice for the viscous dissipation is given by a combination of a classical Newtonian flow and a simple multipolar dissipation model, namely

\[
L[v(t), v(t)] := \frac{1}{2} (\text{tr} \varepsilon[v])^2 + \mu \text{tr}(\varepsilon[v])^2 + \gamma |D^m v|^2, \tag{1.1}
\]

where \( \varepsilon[v] = \frac{1}{2} (\nabla v + \nabla v^T) \), \( m > 1 + \frac{d}{2} \) and \( \mu, \gamma > 0 \) (throughout this paper gradient \( \nabla \), divergence \( \text{div} \), and higher order derivatives \( D^m \) are always evaluated with respect to the spatial variables). The first two terms of the integrand represent the usual dissipation density in a Newtonian fluid, whereas the third term represents a higher order measure for friction. Under suitable assumptions on \( L \) it is shown in [13, Theorem 2.5] that paths of finite energy, which connect two diffeomorphisms \( \psi(0) = \psi_A \) and \( \psi(1) = \psi_B \), are indeed one-parameter families of diffeomorphisms. Furthermore, for any sequence of paths of bounded energy a subsequence converges uniformly to an energy minimizing path, in particular the minimizing path solves \( \dot{\psi}(t, \cdot) = v(t, \psi(t, \cdot)) \) for every \( t \in [0,1] \), where \( v \) is the energy minimizing velocity (cf. [13, Theorem 3.1]). Given two image intensity functions \( u_A, u_B : D \to \mathbb{R} \) an associated geodesic path is a family of images \( u = (u(t) : D \to \mathbb{R})_{t \in [0,1]} \) with \( u(0) = u_A \) and \( u(1) = u_B \), which minimizes the path energy. The associated flow of images is given by \( u(t) = u_A \circ \psi^{-1}(t) \). In medical applications [4], the diffeomorphisms represent deformations of anatomic reference structures described by some image \( u_A \). Thus, each diffeomorphism \( \psi(t) : D \to \mathbb{R}^d \) for \( t \in [0,1] \) represents a particular anatomic configuration or shape of these structures. Let us remark that this model is obviously invariant under rigid body motions, i.e. rigid body motions are generated by motion fields \( v \) with spatially constant, skew symmetric Jacobian, for which \( \varepsilon[v] = 0 \) and \( D^m v = 0 \).

**Metamorphosis.** The metamorphosis approach was first proposed by Miller and Younes [24] and comprehensively analyzed by Trouvé and Younes [35]. It allows in addition for image intensity variations along motion paths. Conceptually and under the assumption that the family of images \( u \) is sufficiently smooth, the associated metric for some parameter \( \delta > 0 \) can be written as

\[
g(\dot{u}, \dot{u}) = \min_{v : D \to \mathbb{R}^d} \int_D L[v, v] + \frac{1}{\delta} (\dot{u} + \nabla u \cdot v)^2 \, dx
\]

and induces the path energy \( E[u] = \int_0^1 g(\dot{u}(t), \dot{u}(t)) \, dt \). Here, \( \delta \frac{\partial}{\partial t} u = \dot{u} + \nabla u \cdot v \) denotes the material derivative of \( u \). Obviously, the same temporal change \( \dot{u}(t) \) in the image intensity can be implied by different motion fields \( v(t) \) and different associated material derivatives \( \delta \frac{\partial}{\partial t} u \), i.e. \( \dot{u}(t) = \delta \frac{\partial}{\partial t} u \). In fact, one introduces a nonlinear geometric structure on the space of images by considering equivalence classes of pairs \( (v, \delta \frac{\partial}{\partial t} u) \) as tangent vectors in the space of images, where such pairs are supposed to be equivalent if they imply the same temporal change \( \dot{u} \). Hence, to evaluate the metric on such tangent vectors one has to minimize over the elements of the equivalence class and computing a geodesic path requires to optimize both the temporal change of the image intensity and the motion field. Thereby, the first term \( L[v, v] \) reflects the cost of the underlying transport and the term \( \frac{1}{\delta} (\delta \frac{\partial}{\partial t} u)^2 \) penalizes the variation of the image intensity along motion paths.

However, typically images are not smooth and paths in image space are neither smooth in time nor in space. Thus, the classical notion of the material derivative \( \dot{u} + \nabla u \cdot v \) is not well-
defined. In [34], Trouvè and Younes established a suitable generalization of the above non-linear geometric structure on the space of images \( L^2(D) := L^2(D, \mathbb{R}) \) based on a proper notion of weak material derivatives. Here, we recall the fundamental ingredients of this approach. In fact, for \( v \in L^2((0, 1), W^{m,2}(D, \mathbb{R}^d) \cap W_0^{1,2}(D, \mathbb{R}^d)) \) the function \( z \in L^2((0, 1), L^2(D)) \) is defined as a weak material derivative of a function \( u \in L^2((0, 1), L^2(D)) \) if

\[
\int_0^1 \int_D \eta z \, dx \, dt = - \int_0^1 \int_D (\partial_t \eta + \text{div}(\eta v)) u \, dx \, dt
\]

for \( \eta \in C_0^\infty((0, 1) \times D) \). Here, \( W^{m,2} \) denotes the usual Sobolev space of functions with square-integrable derivatives up to order \( m \), and \( W_0^{1,2} \) is the space of function in \( W^{1,2} \) with vanishing trace on the boundary. In terms of Riemannian manifolds, Trouvè and Younes equipped the space of images \( L^2(D) \) with the following nonlinear structure: Let

\[
N_u = \left\{ w = (v, z) \in W : \int_D z \eta + u \text{div}(\eta v) \, dx = 0 \, \forall \eta \in C_0^\infty(D) \right\}.
\]

For \( W = (W^{m,2}(D, \mathbb{R}^d) \cap W_0^{1,2}(D, \mathbb{R}^d)) \times L^2(D) \) the tangent space at \( u \in L^2(D) \) is defined as \( T_u L^2(D) = \{ u \} \times W / N_u \) and elements in this tangent space, which are equivalence classes, are denoted by \( (u, (v, z)) \). The tangent bundle is given by

\[
TL^2(D) = \bigcup_{u \in L^2(D)} T_u L^2(D).
\]

Furthermore, let \( \pi(u, (v, z)) = u \) be the projection onto the image manifold. Indeed, this is a weak formulation of the above notion of a tangent space as an equivalence class. Following the usual Riemannian manifold paradigm, a curve \( u \in C^0([0, 1], L^2(D)) \) in the space of images is called \emph{continuously differentiable}, if there is a continuous curve \( t \mapsto w(t) = (v(t), z(t)) \) in \( W \) such that for any \( \eta \in C_0^\infty(D) \) the mapping \( t \mapsto \int_D u(t) \eta \, dx \) is continuously differentiable (denoted by \( u \in C^1([0, 1], L^2(D)) \)) and

\[
\frac{d}{dt} \left( \int_D u(t) \eta \, dx \right) = \int_D z(t) \eta + u(t) \text{div}(\eta v(t)) \, dx.
\]

In fact, for a curve \( t \mapsto \gamma(t) = \left( u(t), (v(t), z(t)) \right) \) in \( TL^2(D) \) the function \( z \) is the (weak) material derivative if (1.3) holds for all test functions \( \eta \in C_0^\infty([0, 1] \times D) \) and all times \( t \in (0, 1) \). Furthermore, a curve \( u \in C^0([0, 1], L^2(D)) \) is defined to be \emph{regular} in the space of images (denoted by \( u \in H^1((0, 1), L^2(D)) \)), if there exists a measurable path \( \gamma : [0, 1] \to TL^2(D) \) with \( \pi(\gamma) = u \) and bounded \( L^2 \)-norm in space and time, such that

\[
- \int_0^1 \int_D u \partial_t \eta \, dx \, dt = \int_0^1 \int_D z \eta + u \text{div}(\eta v) \, dx \, dt
\]

for all \( \eta \in C_0^\infty((0, 1) \times D) \). In fact, a continuously differentiable path \( u \in C^1((0, 1), L^2(D)) \) is always regular, i.e. \( u \in H^1((0, 1), L^2(D)) \) (cf. [34, Proposition 4]). Now, for a regular path \( u \in H^1((0, 1), L^2(D)) \) and for the quadratic form \( L[v, v] \) being coercive on \( W^{m,2}(D, \mathbb{R}^d) \cap W_0^{1,2}(D, \mathbb{R}^d) \) (which can be easily verified for \( L \) given in (1.1) using Korn’s Lemma) one can rigorously define the path energy

\[
E[u] = \int_0^1 \inf_{(v, z) \in T_u(t) L^2(D)} \int_D L[v, v] + \frac{1}{2} z^2 \, dx \, dt.
\]
In [34] Trouvé and Younes proved the existence of minimizing paths for given boundary data in time. Adapted to our notion they have shown that for \( m > 1 + \frac{d}{2} \) and \( \gamma, \delta > 0 \) and given images \( u_A, u_B \in L^2(D) \) there exists a curve \( u \in H^1((0,1), L^2(D)) \) with \( u(0) = u_A \) and \( u(1) = u_B \) such that

\[
\mathcal{E}[u] = \inf \{ \mathcal{E}[\tilde{u}] : \tilde{u} \in H^1((0,1), L^2(D)), \tilde{u}(0) = u_A, \tilde{u}(1) = u_B \}.
\]

Moreover, the infimum in (1.5) is attained for all \( t \in [0,1] \), i.e. there exist minimizing \((v, z) \in T_{u(t)}L^2(D)\).

The proof relies on the observation that \( W^{m,2}(D) \cap W^{1,2}_0(D) \) compactly embeds into \( C^{1,\alpha}_0(D) \) for \( \alpha < m - 1 - \frac{d}{2} \). The existence of a geodesic path then follows from [34, Theorem 6], whereas the addendum is a consequence of [34, Theorem 2].

2. The variational time discretization. In what follows, we develop a variational approach for the time discretization of geodesic paths in the metamorphosis model. This will be based on a time discrete approximation of the above time continuous path energy (1.5). To this end, we have to restrict the space of input images slightly. We assume that the input images \( u_A \) and \( u_B \) are bounded and continuous up to a Lebesgue null set, i.e. \( u_A, u_B \in \mathcal{I} \) for

\[
\mathcal{I} = \{ u \in L^\infty(D) : \exists \text{ Lebesgue null set } S(u) \subset D \text{ s.t. } u \text{ is continuous on } D \setminus S(u) \}.
\]

This space contains in particular images which are piecewise continuous with a discontinuity set (edge set) of codimension 1. In what follows, we suppose that \( \gamma, \delta > 0 \), \( m > 1 + \frac{d}{2} \), and define for arbitrary images \( u, \tilde{u} \in \mathcal{I} \) and for a particular energy density \( W \) a discrete energy

\[
\mathcal{W}[u, \tilde{u}] = \min_{\phi \in \mathcal{A}} \int_D W(D\phi) + \gamma |D^m \phi|^2 + \frac{1}{\delta} |\tilde{u} \circ \phi - u|^2 \, dx,
\]

(2.1)

where \( \mathcal{A} \) is the set of admissible deformations. Throughout this paper, we make the following assumptions with regard to the energy density function \( W \):

(W1) \( W \) is non-negative and polyconvex,

(W2) \( W(A) \geq \beta_0 (\det A)^{-s} - \beta_1 \) for \( \beta_0, \beta_1, s > 0 \) and every invertible matrix \( A \) with \( \det A > 0 \), \( W(A) = \infty \) for \( \det A \leq 0 \), and

(W3) \( W \) is sufficiently smooth and the following consistency assumptions with respect to the differential operator \( L \) hold true: \( W(\mathbb{I}) = 0 \), \( DW(\mathbb{I}) = 0 \) and

\[
\frac{1}{2} D^2 W(\mathbb{I})(B, B) = \frac{\lambda}{2} (\text{tr} B)^2 + \mu \text{tr} \left( \frac{(B + B^T)}{2} \right)^2 \quad \forall B \in \mathbb{R}^{d,d}.
\]

Furthermore, the set of admissible deformations is given as

\[
\mathcal{A} = \{ \phi \in W^{m,2}(D, D) : \det D\phi > 0 \text{ a.e. in } D, \phi = \mathbb{I} \text{ on } \partial D \}.
\]

In what follows, we use the symbol \( \mathbb{I} \) both for the identity mapping \( x \mapsto x \) and the identity matrix. The first two assumptions ensure the existence of a minimizing deformation in (2.1) and thus the well-posedness of the discrete energy \( \mathcal{W}[u, \tilde{u}] \) for \( u, \tilde{u} \in \mathcal{I} \). Note that [3, Theorem 1] already implies the global invertibility (a.e.) of every \( \phi \in \mathcal{A} \) because \( \mathcal{A} \subset W^{1,p}(D) \) for \( p > d \). The third assumption states that the definition of \( \mathcal{W} \) is consistent with the underlying dissipation described by the quadratic form \( L \).
Now, we consider discrete curves $u = (u_0, \ldots, u_K) \in \mathcal{I}^{K+1}$ in image space and define a *discrete path energy* as the sum over pairwise matching functionals $\mathcal{W}$ evaluated on consecutive images of these discrete curves as follows

$$E_K[u] := K \sum_{k=1}^{K} W[u_{k-1}, u_k].$$

(2.2)

We refer to [29] for the introduction of such a variational time discretization on shape manifolds. Based on this path energy we can define *discrete geodesics* as follows.

**Definition 2.1.** Let $u_A, u_B \in \mathcal{I}$ and $K \geq 1$. A discrete geodesic connecting $u_A$ and $u_B$ is a discrete curve in image space which minimizes $E_K$ over all discrete curves $u = (u_0, \ldots, u_K) \in \mathcal{I}^{K+1}$ with $u_0 = u_A$ and $u_K = u_B$.

Due to the assumption (W3) the energy on the right-hand side of (2.1) scales quadratically in the displacement $\phi - 1$, which itself is expected to scale linearly in the time step $\tau = \frac{1}{K}$. This already motivates the coefficient $K$ in front of the discrete path energy. For the rigorous justification, we refer to the proof of Theorem 4.1 on the $\Gamma$-convergence estimates.

In general, we want the energy density to fulfill two desirable properties: isotropy and rigid body motion invariance. A suitable choice for an isotropic and rigid body motion invariant energy density $W$ in the case $d = 2$, which fulfills the assumptions (W1-3) is given by

$$W(D\phi) = a_1 (\text{tr}(D\phi^T D\phi))^q + a_2 (\det D\phi)^r + a_3 (\det D\phi)^{-s} + a_4$$

(2.3)

with $q, r, s \geq 1$ and coefficients $a_1 = \frac{2-q}{q} \mu$, $a_2 = \frac{\lambda + \mu - \mu q - \mu s}{rs + q s}$, $a_3 = \frac{\lambda + \mu - \mu q + \mu r}{rs + q s}$, and $a_4 = \frac{\mu (q^2 - rs - q(1+r-s)) - \lambda q}{qr}$, which is a special case of an Ogden material. Indeed, it is possible to choose for given $\lambda, \mu > 0$ the parameters $q, r, s$ in such a way that the resulting coefficients $a_1, a_2$ and $a_3$ are positive. For the definition of a corresponding energy density in the case $d = 3$ we refer to [11, Section 4.9/4.10].

In the discrete path energy, two opposing effects can be observed. For a given discrete curve $u$ and (minimizing) deformations $\phi_1, \ldots, \phi_K$, the last term penalizes intensity variations along the discrete motion path $(x, \phi_1(x), (\phi_2 \circ \phi_1)(x), \ldots, (\phi_K \circ \ldots \circ \phi_1)(x))$, whereas the first two terms penalize deviations of the (discrete) flow along these discrete motion paths from rigid body motions. We will see that $K(u_k \circ \phi_k - u_{k-1})$ reflects a time discrete material derivative along the above discrete motion path, whereas the first two terms represent a discrete dissipation density.

3. *Well-posedness of the discrete path energy and existence of discrete geodesics.*

In this section, we will show that for images $u, \tilde{u} \in \mathcal{I}$ a minimizing deformation in the definition of $\mathcal{W}[u, \tilde{u}]$ exists, which renders the definition of the discrete path energy well-posed. Furthermore, we will prove existence of a minimizing path $u$ of the discrete path energy $E_K$ and thereby establish the existence of a discrete geodesic.

**Proposition 3.1 (Well-posedness of $\mathcal{W}$).** Under the above assumptions (W1-2) and for $u, \tilde{u} \in \mathcal{I}$ there exists a deformation $\psi \in \mathcal{A}$ depending on $u$ and $\tilde{u}$ such that $\mathcal{W}[u, \tilde{u}] = \mathcal{W}^D[u, \tilde{u}, \psi]$, where

$$\mathcal{W}^D[u, \tilde{u}, \psi] := \int_D W(D\psi) + \gamma |D^m \psi|^2 + \frac{1}{\delta} |\psi - u|^2 d x.$$

Moreover $\psi$ is a diffeomorphism and $\psi^{-1} \in C^{1, \alpha}(D)$ for $\alpha \in (0, m - 1 - \frac{d}{2})$.

**Proof.** The proof proceeds in four steps. Note that in what follows we will make use of the summation convention.
**Step 1.** Due to (W1) we know that $0 \leq \underline{W} := \inf_{\phi \in \mathcal{A}} \mathcal{W}^D[u, \bar{u}, \phi]$ and since $I \in \mathcal{A}$ we have that $\mathcal{W}^D[u, \bar{u}, I] < \infty$. Consider a minimizing sequence $(\phi^j)_{j \in \mathbb{N}} \subset \mathcal{A}$ with monotonously decreasing energy $\mathcal{W}^D[u, \bar{u}, \phi^j] < \infty$, which converges to $\underline{W}$. In particular, $\underline{W} = \mathcal{W}^D[u, \bar{u}, \phi^1] < \infty$ is an upper bound. As a consequence of Korn’s inequality and the Gagliardo-Nirenberg inequality for bounded domains (see [27, Theorem 1]), we can deduce that the minimizing sequence is bounded in $W^{m,2}(D)$. Hence, due to the reflexivity of this space, there is a weakly convergent subsequence in $W^{m,2}(D)$, again denoted by $\phi^j \rightharpoonup \phi$, and by the Sobolev embedding theorem we can assume uniform convergence of $\phi^j \to \phi$ in $C^{1,\alpha}(D)$ for $\alpha \in (0, m - 1 - \frac{3}{2})$.

**Step 2.** We show that the deformation $\phi$ belongs to $\mathcal{A}$. To this end, we will control the measure of the set $S_\epsilon = \{ x \in D \mid \det D\phi \leq \epsilon \}$ for sufficiently small $\epsilon > 0$. Indeed, by using (W1), (W2) and Fatou’s lemma, we obtain
\[
\beta_0 \epsilon^{-s} |S_\epsilon| \leq \beta_0 \int_{S_\epsilon} (\det D\phi)^{-s} \, dx \leq \int_{S_\epsilon} W(D\phi) \, dx + \beta_1 |D| \leq \underline{W} + \beta_1 |D|
\]
and thus $|S_\epsilon| \leq \frac{(\underline{W} + \beta_1 |D|) \epsilon^s}{\beta_0}$, which shows $|S_0| = 0$ and $\det D\phi > 0$ a.e. on $D$. This implies $\phi \in \mathcal{A}$ (note $\phi \in W^{1,p}$ for a $p > d$) and due to [3, Theorem 1] and $\phi \in W^{m,2}(D)$ the deformation $\phi$ is injective and a homeomorphism. By Sard’s theorem for Hölder spaces (cf. [7]) we additionally know $\phi^{-1} \in C^{1,\alpha}(D)$.

**Step 3.** Due to our assumptions on the image space $I$ the discontinuity set $S(\bar{u})$ of the image $\bar{u}$ is a Lebesgue null set. Thus, for given $\rho > 0$ there exists an open set $M^\rho \subset D$ such that $S(\bar{u}) \subset M^\rho$ and $|M^\rho| \leq \rho$. Additionally, using [3, Theorem 1 (ii)] we can deduce the estimate
\[
|\phi^{-1}(M^\rho) \setminus S_\epsilon| \leq \frac{1}{\epsilon} \int_{\phi^{-1}(M^\rho)} \det D\phi \, dx = \frac{1}{\epsilon} \int_{M^\rho} \, dx \leq \frac{\rho}{\epsilon},
\]
which allows us to control the preimage of $M^\rho$ with respect to $\phi$ restricted to $D \setminus S_\epsilon$.

**Step 4.** Let $j(\epsilon) \in \mathbb{N}$ be such that $\mathcal{W}^D[u, \bar{u}, \phi^j] \leq \mathcal{W}^D[u, \bar{u}, \phi^{j(\epsilon)}] \leq \underline{W} + \epsilon$ for all $j \geq j(\epsilon)$. The measure of the set $R_{\rho,\epsilon} := \phi^{-1}(M^\rho) \cup S_\epsilon$ can be estimated as follows
\[
|R_{\rho,\epsilon}| \leq \frac{\rho}{\epsilon} + \frac{(\underline{W} + \beta_1 |D|) \epsilon^s}{\beta_0}.
\]
Moreover, the sequence $(\tilde{u} \circ \phi^j - u)_{j \in \mathbb{N}}$ converges pointwise a.e. to $\tilde{u} \circ \phi - u$ on $D \setminus R_{\rho,\epsilon}$. By Lebesgue convergence theorem, we can enlarge $j(\epsilon)$ if necessary such that for all $j \geq j(\epsilon)$
\[
\left| \int_{D \setminus R_{\rho,\epsilon}} |\tilde{u} \circ \phi^j(x) - u(x)|^2 - |\tilde{u} \circ \phi(x) - u(x)|^2 \, dx \right| \leq \epsilon.
\]
Let $C := 2(|\tilde{u}|^2_{L^\infty(D)} + |u|^2_{L^\infty(D)})$. Again, using (W1), (W2) and Fatou’s lemma we infer
\[
\mathcal{W}^D[u, \bar{u}, \phi] = \int_D W(D\phi) + \gamma |D^m\phi|^2 + \frac{1}{\delta} |\tilde{u} \circ \phi - u|^2 \, dx \leq \liminf_{j \to \infty} \int_D W(D\phi^j) + \gamma |D^m\phi^j|^2 \, dx + \int_{D \setminus R_{\rho,\epsilon}} \frac{1}{\delta} |\tilde{u} \circ \phi^j - u|^2 \, dx + \frac{C}{\delta} |R_{\rho,\epsilon}|.
\]
\[
\begin{align*}
&\leq \liminf_{j \to \infty} \int_D W(D\phi^j) + \gamma|D^m \phi^j|^2 \, dx \\
&+ \liminf_{j \to \infty} \int_{D \cap R_{p,e}} \frac{1}{\delta} \tilde{u} \circ \phi^j - u^j \, dx + \frac{\epsilon}{\delta} + \frac{C'}{\delta} |R_{p,e}| \\
&\leq \liminf_{j \to \infty} \int_D W(D\phi^j) + \gamma|D^m \phi^j|^2 + \frac{1}{\delta} \tilde{u} \circ \phi^j - u^j \, dx + \frac{\epsilon}{\delta} + \frac{2C'}{\delta} |R_{p,e}| \\
&\leq W + \epsilon + \frac{\epsilon}{\delta} + \frac{2C'}{\delta} |R_{p,e}|.
\end{align*}
\]

Taking into account the above estimate for \(R_{p,e}\) we deduce that the term \(\epsilon + \frac{\epsilon}{\delta} + \frac{2C'}{\delta} |R_{p,e}|\) can be made arbitrarily small, which proves the claim. \(\square\)

Next, for a given discrete path \(u = (u_0, \ldots, u_K) \in \mathcal{I}^{K+1}\) we define a discrete path energy explicitly depending on a \(K\)-tuple of deformations \(\Phi = (\phi_1, \ldots, \phi_K) \in \mathcal{A}^K\) as follows:

\[
E^D_K[u, \Phi] := K \sum_{k=1}^K W^D[u_{k-1}, u_k, \phi_k].
\]

As an immediate consequence of Proposition 3.1, there exists a vector of deformations \(\Phi \in \mathcal{A}^K\) such that \(E^D_K[u, \Phi] = E_K[u]\). If the images \(u_0, \ldots, u_K\) are sufficiently smooth, the corresponding system of Euler-Lagrange equations for \(\phi_k\) is given by

\[
\int_D W_A(D\phi_k) : D\theta + 2\gamma D^m \phi_k : D^m \theta + \frac{2}{\delta}(u_k \circ \phi_k - u_{k-1})(\nabla u_k \circ \phi_k) \cdot \theta \, dx = 0
\]

for all \(1 \leq k \leq K\) and all test deformations \(\theta \in W^{m,2}(D, \mathbb{R}^d) \cap W^{1,2}_0(D, \mathbb{R}^d)\), which is a system of nonlinear PDEs of order \(2m\). Here “:” denotes the sum over all pairwise products of two tensors.

Before we discuss the existence of discrete geodesics, we first present the following partial result, which can be regarded as a counterpart of Proposition 3.1 because it establishes the existence of an energy minimizing vector of images \(u\) for a given vector of deformations \(\Phi\).

**Proposition 3.2.** Let \(u_A, u_B \in \mathcal{I}\) and \(K \geq 2\). Assume a vector \(\Phi \in \mathcal{A}^K\) is given. Then, there exists \(u = (u_0, \ldots, u_K) \in \mathcal{I}^{K+1}\) with \(u_0 = u_A, u_K = u_B\) such that

\[
E^D_K[u, \Phi] = \inf_{\tilde{u} \in \mathcal{I}^{K+1}, \tilde{u}_0 = u_A, \tilde{u}_K = u_B} E^D_K[\tilde{u}, \Phi].
\]

Furthermore, the estimate \(\|u_k\|_{L^\infty(D)} \leq \max\left(\|u_A\|_{L^\infty(D)}, \|u_B\|_{L^\infty(D)}\right)\) holds for all \(k = 1, \ldots, K - 1\).

**Proof.** Let \(\tilde{u} = (u_0^j, \ldots, u_{K-1}^j) \subset \mathcal{I}^{K-1}\) be a minimizing sequence for the energy \(E^D_K(u_A, u_B, \Phi)\). Obviously, the energy is reduced by a truncation of the images, i.e. replacing \(u_k^j\) by \(\max(\min(u_k^j, \tilde{u}), -\tilde{u})\) with \(\tilde{u} = \max(\|u_A\|_{L^\infty(D)}, \|u_B\|_{L^\infty(D)})\). Hence, we can assume that \(\|u_k^j\|_{L^\infty(D)} \leq \tilde{u}\) for all \(j \in \mathbb{N}\). Thus, there exists a weakly convergent subsequence in \(L^2(D)^{K-1}\) with weak limit \(V = (u_1, \ldots, u_{K-1})\) and \(\|u_k\|_{L^\infty(D)} \leq \tilde{u}\).

We still have to show \(u_k \in \mathcal{I}\) for \(k = 1, \ldots, K - 1\). To this end we take into account the transformation rule

\[
\begin{align*}
\int_D (u_k \circ \phi_k - u_{k-1})^2 + (u_{k+1} \circ \phi_{k+1} - u_k)^2 \, dx \\
= \int_D (u_k - u_{k-1} \circ \phi_k^{-1})^2 (\det D\phi_k)^{-1} \circ \phi_k^{-1} + (u_{k+1} \circ \phi_{k+1} - u_k)^2 \, dx
\end{align*}
\]
and derive from the Euler-Lagrange equation $\partial_u E^D_K[u, \Phi] = 0$ the pointwise condition
\[
\left( (u_k - u_{k-1} \circ \phi_k^{-1}) \left( (\det D\phi_k)^{-1} \circ \phi_k^{-1} \right) + (u_k - u_{k+1} \circ \phi_{k+1}) \right) (x) = 0 \text{ for a.e. } x \in D,
\]
which can also be written as
\[
u_k(x) = \frac{u_{k+1} - u_k \circ \phi_{k+1}(x) + (u_k - u_{k-1} \circ \phi_k^{-1}(x))(\det D\phi_k)^{-1} \circ \phi_k^{-1}(x)}{1 + (\det D\phi_k)^{-1} \circ \phi_k^{-1}(x)}
\]
for a.e. $x \in D$. This leads to a linear system of equations for $(u_1, \ldots, u_{K-1})$, where evaluations at deformed positions are combined with evaluations at non-deformed positions, which we consider as a block tridiagonal operator equation. In fact, defining for each $x \in D$ the discrete transport path $X(x) = (X_0(x), X_1(x), X_2(x), \ldots, X_K(x))^T \in \mathbb{R}^{K+1}$ with $X_0(x) = x$ and $X_k(x) = \phi_k(X_{k-1}(x))$ for $k \in \{1, \ldots, K\}$ and the vector of associated intensity values
\[
U(\hat{u}, \Phi)(x) := (u_1(X_1(x)), u_2(X_2(x)), \ldots, u_{K-1}(X_{K-1}(x)))^T \in \mathbb{R}^{K-1}
\]
we obtain for $K \geq 3$ and a.e. $x \in D$ a linear system of equations
\[
A[\Phi](x)U(\hat{u}, \Phi)(x) = R(\Phi)(x)
\]
on $\mathbb{R}^{K-1}$. In this case $A[\Phi](x) \in \mathbb{R}^{K-1,K-1}$ is a tridiagonal matrix with
\[
A[\Phi](x)_{k,k+1} = -\frac{1}{1 + (\det D\phi_k)^{-1} \circ \phi_k^{-1}(X_k(x))}, \quad A[\Phi](x)_{k,k} = 1 + 1 + (\det D\phi_k)^{-1} \circ \phi_k^{-1}(X_k(x)), \quad A[\Phi](x)_{k,k-1} = -\frac{1}{1 + (\det D\phi_k)^{-1} \circ \phi_k^{-1}(X_k(x))},
\]
and $R[\Phi](x) \in \mathbb{R}^{K-1}$ is given by
\[
R[\Phi](x) = \left( u_A(X_1(x)) \frac{(\det D\phi_1)^{-1}(x)}{1 + (\det D\phi_1)^{-1}(x)}, 0, \ldots, 0, \frac{u_B(X_K(x))}{1 + (\det D\phi_K)^{-1}(X_{K-2}(x))} \right)^T.
\]
For any vector of regular deformations $\Phi \in A^K$ we recall that $\det D\phi_k > 0$ for $k = 1, \ldots, K$ and $\Phi \in (C^1(D))^K$. From this we deduce that for a.e. $x \in D$ the matrix $A[\Phi](x)$ is irreducibly diagonally dominant, which implies invertibility. Thus, for all $x \in D$ there exists a unique solution $U(\hat{u}, \Phi)(x)$ solving (3.3). From $U(\hat{u}, \Phi)(x) = (A[\Phi](x))^{-1} R[\Phi](x)$ for all $x \in D$ we infer that $u_k$ is continuous on $D \setminus S_k$ with $S_k := X_k \left( S(u_A) \cup X_k^{-1}(S(u_B)) \right)$ and $S_k$ is a null set. Hence, $u_k \in T$ for all $k = 1, \ldots, K - 1$.

Finally, let us note that the $L^\infty$-estimate also follows immediately from the fact that $u_k(X_k(x))$ can be written as a convex combination of $u_{k-1}(X_{k-1}(x))$ and $u_{k+1}(X_{k+1}(x))$ for $k = 1, \ldots, K - 1$ due to (3.1).

**Remark 3.3.** (i) If the input images $u_A$ and $u_B$ are in $C^{0,\alpha}(D)$ for $\alpha \leq m - 1 - \frac{d}{2}$, then the proof of Theorem 3.2 also shows that $u_k \in C^{0,\alpha}(D)$ for all $k = 1, \ldots, K - 1$. (ii) The intensity values along the discrete transport path $X(x)$ depend in a unique way on the values at the two end points $x$ and $X_K(x)$ and each $u_k(X_k(x))$ is a weighted average of the intensities $u_A(x)$ and $u_B(X_K(x))$, where the weights reflect the compression and expansion associated with the deformations along the discrete transport paths.

Now, we are in the position to prove the existence of discrete geodesics making use of the existence of a minimizing family of deformations for the energy $E^D$ and a given discrete
image path as a consequence of Proposition 3.1 and the existence of an optimal discrete image path for a given family of deformations as stated in Proposition 3.2.

**Theorem 3.4 (Existence of discrete geodesics).** Let \( u_A, u_B \in \mathcal{I} \) and \( K \geq 2 \). Then there exists \( \hat{u} \in \mathcal{T}^{K-1} \) such that \( \mathbf{E}_K([u_A, \hat{u}, u_B]) = \inf_{\hat{v} \in \mathcal{T}^{K-1}} \mathbf{E}_K([u_A, \hat{v}, u_B]) \).

**Proof.** Let us assume that \((\hat{u}^j)_{j \in \mathbb{N}} \in \mathcal{T}^{K-1} \) with \( \hat{u}^j = (u_{i^j_1}, \ldots, u_{i^j_{K-1}}) \) is a minimizing sequence of the discrete path energy \( \mathbf{E}_K([u_A, \hat{v}, u_B]) \), where \( \mathbf{E}_K \) is an upper bound of the discrete path energy. Due to Proposition 3.1, for every \( \hat{u}^j \) there exists a family of optimal deformations \( \Phi^j = (\phi^j_1, \ldots, \phi^j_K) \in \mathcal{A}^K \) with \( \mathbf{E}_K^j([u_A, \hat{v}^j, u_B], \Phi^j) \leq \mathbf{E}_K([u_A, \hat{u}^j, u_B], \Phi^j) \) for all \( \Phi' \in \mathcal{A}^K \). Furthermore, we can assume (by possibly replacing \( \Phi^j \) and thereby further reducing the energy) that \( \hat{v}^j \) already minimizes the discrete path energy \( \mathbf{E}_K([u_A, \hat{v}, u_B], \Phi^j) \) over all \( \hat{v} \in \mathcal{T}^{K-1} \). From the proof of Proposition 3.2 we know that \( \hat{v}^j(x) \) is obtained solving the set of linear system of equations \( A[\Phi^j](x)U(\hat{u}^j, \Phi^j)(x) = R[\Phi^j](x) \) for all \( x \in D \) and the resulting images \( u_{i^j_k} \) are uniformly bounded for \( k = 1, \ldots, K-1 \), i.e., \( \|u_{i^j_k}\|_\infty \leq \max\{\|u_A\|_{L^\infty(D)}, \|u_B\|_{L^\infty(D)}\} \) for all \( k \in \mathbb{N} \). Hence, a subsequence of \((u_{i^j_k})_{j \in \mathbb{N}} \) converging weak-* in \( L^\infty(D) \) to some \( u_k \). Next, we note that due to the coercivity estimate \( \|D^m \phi^j_k\|_2^2 \leq \mathbf{E}_K^j \) and the Gagliardo-Nirenberg inequality the deformations \( \phi^j_k \) are uniformly bounded in \( W^{m,2}(D, \mathbb{R}^d) \) for \( k = 1, \ldots, K \). Together with the compact embedding of \( W^{m,2}(D, \mathbb{R}^d) \) into \( C^{1,\alpha}(D, \mathbb{R}^d) \) for \( 0 < \alpha < m - 1 - \frac{d}{2} \), this implies that (up to the selection of another subsequence) \( \Phi^j \) converges to \( \Phi = (\phi_1, \ldots, \phi_K) \) weakly in \( (W^{m,2}(D, \mathbb{R}^d))^K \) and uniformly in \( (C^{1,\alpha}(D, \mathbb{R}^d))^K \). Following the same line of arguments as in Step 2 of the proof of Proposition 3.1, we in addition infer that \( \det D\phi_k > 0 \) a.e. in \( D \) for \( k = 1, \ldots, K \) and thus \( \Phi \in \mathcal{A}^K \). Finally, we argue from the strong convergence of \( \Phi^j \) in \( (C^{1,\alpha}(D, \mathbb{R}^d))^K \) that \( \hat{u}^j \) converges weak-* to \( \hat{u}(x) = (A[\Phi^j](x))^{-1}R[\Phi^j](x) \) for all \( x \in D \) and \( \hat{u} = (u_1, \ldots, u_{K-1}) \). This implies

\[
\sum_{k=1}^K \int_D (u_k \circ \phi_k - u_{k-1})^2 \, dx = \liminf_{j \to \infty} \sum_{k=1}^K \int_D (u_{i^j_k} \circ \phi^j_k - u_{i^j_{k-1}})^2 \, dx.
\]

Together with the weak lower semi-continuity of \( \phi \mapsto \int_D W(D\phi) + \gamma |D^m \phi|^2 \, dx \) one achieves

\[
\mathbf{E}_K[u_A, \hat{u}, u_B] = \mathbf{E}_K^j([u_A, \hat{u}^j, u_B], \Phi) \leq \liminf_{j \to \infty} \mathbf{E}_K^j([u_A, \hat{u}^j, u_B], \Phi^j) = \liminf_{j \to \infty} \mathbf{E}_K[u_A, \hat{u}^j, u_B],
\]

which proves the claim. \( \square \)

**4. Convergence of discrete geodesic paths.** In what follows, we will study the convergence of minimizers of our discrete variational model (2.2) for \( K \to \infty \) to minimizers of the continuous model (1.5) and thus the convergence of discrete geodesic paths to continuous geodesic paths. To this end, we prove \( \Gamma \)-convergence estimates for a natural extension of the discrete path energy. For an introduction to \( \Gamma \)-convergence, we refer to [12].

At first, let us discuss a suitable interpolation of continuous paths. For fixed \( K \geq 2 \) and time step size \( \tau = \frac{1}{K} \) let \( t_k = k\tau \) denote the time step corresponding to a vector of images \( u = (u_0, \ldots, u_K) \in \mathcal{T}^{K+1} \). For a vector \( \Phi = (\phi_1, \ldots, \phi_K) \in \mathcal{A}^K \) of optimal deformations resulting from the minimization in (2.1) we define for \( k = 1, \ldots, K \) the motion field \( \mathbf{v}_k = K(\phi_k - I) \) and the induced transport map \( y_k(t, x) = x + (t - \mathbf{v}_k) \) with \( t \in [t_{k-1}, t_k] \). Note that \( y_k(t_{k-1}, x) = x \) and \( y_k(t_k, x) = \phi_k(x) \). If one assumes
that $\|D\phi_k - 1\|_\infty := \sup_{x \in D} \max_{|v|=1} \left| (D\phi(x) - 1)v \right| < 1$, then $y_k(t, \cdot) = 1 + K(t - t_{k-1})(\phi_k - 1)$ is invertible. Thus, denoting the inverse of $y_k(t, \cdot)$ by $x_k(t, \cdot)$ one obtains the image interpolation $u = U_K[u, \Phi]$ with

$$U_K[u, \Phi](t, x) = u_{k-1}(x_{k-1}(t, x)) + K(t-t_{k-1})(u_k \circ \phi_k - u_{k-1})(x_k(t, x))$$  \hspace{1cm} (4.1)

for $t \in [t_{k-1}, t_k]$. This interpolation represents on each interval $[t_{k-1}, t_k]$ the blending between the images $u_{k-1} = U_K[u, \Phi](t_{k-1}, \cdot)$ and $u_k = U_K[u, \Phi](t_k, \cdot)$ along affine transport paths

$$\{(t, y_k(t, x)) \mid t \in [t_{k-1}, t_k]\}$$

for $x \in D$. Based on this interpolation a straightforward extension $E_K : L^2([0, 1] \times D) \to [0, \infty)$ of the discrete path energy $E_K$ is given by

$$E_K[u] = \begin{cases} E^D_K[u, \Phi] & \text{if } u = U_K[u, \Phi] \text{ with } u \in I^{K+1} \\
+\infty & \text{else} \end{cases}$$

if $\Phi$ is a minimizer of $E^D_K[u, \cdot]$ over $A^K$.

Now, we are in the position to discuss the $\Gamma$-convergence estimates. The next theorem does not explicitly state $\Gamma$-convergence of $E_K$ to $E$. In fact, the underlying spaces slightly vary to deal with the restriction that the solution of the pairwise matching problems requires the underlying images to be in $I$. Nevertheless, the statements of the theorem are sufficient to prove that subsequences of discrete geodesics converge to a continuous geodesic (cf. Theorem 4.2).

**Theorem 4.1 ($\Gamma$-convergence estimates).** **Under the assumptions (W1-3) the estimate**

$$\liminf_{K \to \infty} E_K[u^K] \geq E[u]$$

**holds for every sequence** $(u^K)_{K \in \mathbb{N}} \subseteq L^2([0, 1] \times D)$ **with** $u^K \rightharpoonup u$ **(weakly)** in $L^2([0, 1] \times D)$. **Furthermore for** $u \in L^2([0, 1] \times D)$ **with** $u(t, \cdot) \in I$ **for every** $t \in [0, 1]$ **there exists a sequence** $(u^K)_{K \in \mathbb{N}} \subseteq L^2([0, 1] \times D)$ **with** $u^K \to u$ **in** $L^2([0, 1] \times D)$ **such that the estimate**

$$\limsup_{K \to \infty} E_K[u^K] \leq E[u]$$

**holds.**

**Proof.** Throughout the proof we will use a generic constant $C$ independent of $K$.

[limit estimate] Let $(u^K)_{K \in \mathbb{N}} \subseteq L^2([0, 1] \times D)$ be any sequence of images that converges weakly in $L^2([0, 1] \times D)$ to $u \in L^2([0, 1] \times D)$. To exclude trivial cases, i.e. $\liminf_{K \to \infty} E_K[u^K] = \infty$, we may assume $E_K[u^K] \leq C < \infty$ for all $K \in \mathbb{N}$, which implies $u^K = U_K[u^K, \Phi^K]$ for $u^K = (u^K_0, \ldots, u^K_K) \in I^{K+1}$ and an associated vector of deformations $\Phi^K = (\phi^K_0, \ldots, \phi^K_K)$, which is defined as a vector of (not necessarily unique) solutions of the pairwise matching problems (2.1). Each $\Phi^K$ generates on each time interval $[t_{k-1}, t_k]$ affine transport paths with motion velocity $\dot{\phi}_K(t, y) = K(\phi^K_k - 1)(x^K_k(t, y))$. Here, we use the notation $t_k = \frac{k}{K}$ (for the sake of brevity without explicit reference to the sequence index $K$) and $x^K_k$ is the above defined pullback associated with the deformation $\phi^K_k$ on the interval $[t_{k-1}, t_k]$. It is straightforward to see that for sufficiently large $K$ the one parameter family of images $u^K$ has a classical material derivative

$$z^K(t, y) = \frac{d}{ds} u^K(t + s, y + s\dot{\phi}_K(t, y))|_{s=0} = K (u^K_k \circ \phi^K_K - u^K_{k-1})(x^K_k(t, y))$$  \hspace{1cm} (4.2)

for $t \in (t_{k-1}, t_k)$ and for $k = 1, \ldots, K$. Hence, the regularity of $\Phi^K$ stated in Proposition 3.1 implies that $z^K$ fulfills the equation for the weak material derivative (1.2), i.e.

$$\int_D \int_0^1 z^K \vartheta \, dt \, dx = - \int_D \int_0^1 (\partial_t \vartheta + \text{div}(\dot{\phi}_K \vartheta)) u^K \, dt \, dx$$  \hspace{1cm} (4.3)
for all \( \vartheta \in W^{1,2}_0((0,1) \times D) \) and with \( \dot{v}^K(t,y) = \dot{v}_k^K(t,y) \) for \( t \in [t_k-1, t_k) \). Let us remark that \( \dot{v}^K(t,\cdot) \) vanishes on the boundary \( \partial D \) for \( t \in (0,1) \), which corresponds to the assumption on the continuous velocity \( v \) in the metamorphosis model from the introduction. As a next step we show

$$
\lim_{K \to \infty} \int_D \int_0^1 |z^K|^2 \, dt \, dx = \lim_{K \to \infty} K \sum_{k=1}^{K} \int_D |u^K_k \circ \phi^K_k - u^K_{k-1}|^2 \, dx. \quad (4.4)
$$

Indeed, using (4.2) one obtains

$$
\int_D \int_{t_k-1}^{t_k} |z^K|^2 \, dt \, dx = \int_D \int_{t_k-1}^{t_k} K^2 \left( (u^K_k \circ \phi^K_k - u^K_{k-1}) (x^K(t,x)) \right)^2 \, dt \, dx
$$

$$
= \int_D \int_{t_k-1}^{t_k} K^2 \left( (u^K_k \circ \phi^K_k - u^K_{k-1}) (x) \right)^2 \, \det D\gamma^K_k(t,x) \, dt \, dx,
$$

where \( \det D\gamma^K_k(t,x) = 1 + K(t - t_{k-1})(D\phi^K_k(x) - 1) \). From the uniform bound on the energy we deduce

$$
\sum_{k=1}^{K} \int_D K(u^K_k \circ \phi^K_k - u^K_{k-1})^2 \, dx \leq \delta^2. \quad (4.5)
$$

Furthermore, we can estimate

$$
\| \det \left( I + K(t - t_{k-1})(D\phi^K_k - 1) \right) \|_{L^\infty((0,1) \times D)} \leq C \| \phi^K_k - 1 \|_{C^1(D)}. \nonumber
$$

The Sobolev estimate \( \| \phi - 1 \|_{C^{1,\alpha}(D)} \leq C \| \phi - 1 \|_{W^{m,2}(D)} \) for \( \alpha \leq m - 1 - \frac{d}{2} \) and the Gagliardo-Nirenberg interpolation inequality \( \| \phi - 1 \|_{W^{m,2}(D)} \leq C \| D^m \phi \|_{L^2(D)} \) (cf. [27]) for \( \phi \in W^{m,2}(D, \mathbb{R}^d) \cap W^{1,2}_0(D, \mathbb{R}^d) \) imply

$$
\| \phi^K_k - 1 \|^2_{C^{1,\alpha}(D)} \leq \sum_{l=1}^{K} C \| D^m \phi^K_k \|^2_{L^2(D)} \leq C \frac{\delta^2}{\gamma K}. \quad (4.6)
$$

Together with (4.5) this proves (4.4). Next, from (4.5) and (4.4) we deduce that the material derivatives \( z^K \) are uniformly bounded in \( L^2((0,1) \times D) \) independent of \( K \). Thus, there exists a subsequence, again denoted by \( (z^K)_{K \in \mathbb{N}} \), which converges weakly in \( L^2((0,1) \times D) \) to some \( z \in L^2((0,1) \times D) \) as \( K \to \infty \). By the lower semicontinuity of the \( L^2 \) norm one achieves

$$
\int_D \int_0^1 |z|^2 \, dt \, dx \leq \lim_{K \to \infty} \int_D \int_0^1 |z^K|^2 \, dt \, dx.
$$

Now, we will prove that there exists a velocity field \( v \in L^2((0,1), W^{1,2}_0(D) \cap W^{m,2}(D)) \) such that \( (v, z) \in T_u L^2 \) and

$$
\int_0^1 \int_D L[v,v] \, dx \, dt \leq \liminf_{K \to \infty} K \sum_{k=1}^{K} \int_D W(D\gamma^K_k) + \gamma |D^m \phi^K_k|^2 \, dx.
$$

The second order Taylor expansion around \( t_k-1 \) of the function \( t \mapsto W(1 + (t - t_{k-1})Dv^K_k) \) at \( t = t_k \) gives

$$
W(D\gamma^K_k) = W(1) + \frac{1}{K} DW(1)(Dv^K_k) + \frac{1}{2K^2} D^2 W(1)(Dv^K_k, Dv^K_k) + O(K^{-3}|Dv^K_k|^3)
$$

$$
= \frac{1}{K^3} \left( \frac{m}{2} \text{tr}[v^K_k]^2 + \mu \text{tr}[(v^K_k)^2] \right) + O(K^{-3}|Dv^K_k|^3)
$$

$$
= \frac{1}{K^3} \left( \frac{m}{2} \text{tr}[v^K_k]^2 + \mu \text{tr}[(v^K_k)^2] \right) + O(K^{-3}|Dv^K_k|^3)
$$
with $v^K_k(x) = K(\phi^K_k(x) - x)$. The second equality follows from (W3). Then
\[
K \sum_{k=1}^{K} \int_D W(D\phi^K_k) + \gamma |D^m\phi^K_k|^2 \, dx 
\leq \frac{1}{K} \sum_{k=1}^{K} \int_D \frac{\lambda}{2} (\text{tr}[v^K_k])^2 + \mu \text{tr}(\varepsilon[v^K_k])^2 + \gamma |D^m v^K_k|^2 \, dx + C \sum_{k=1}^{K} \int_D K^{-1} |Dv^K_k|^3 \, dx.
\]

The last term is of order $K^{-\frac{3}{2}}$, which follows from the boundedness of the energy and by applying (4.6), i.e.
\[
\sum_{k=1}^{K} K^{-3} |Dv^K_k|^3 \, dx \leq C \max_{k=1,...,K} \|\phi^K_k - 1\|_{C^1(D)} \sum_{k=1}^{K} \|\phi^K_k - 1\|_{W^{m,2}(D)}^2 \leq CK^{-\frac{3}{2}}.
\]

Next, for $K \to \infty$ the limes inferior of the remainder can be estimated as follows. We define $v^K \in L^2((0,1) \times D)$ via $v^K(t,\cdot) = v^K_k$ for $t \in [t_{k-1}, t_k)$. Due to the uniform bound of the discrete path energy $v^K$ is uniformly bounded in $L^2((0,1), W^{m,2}(D))$ and up to the selection of a subsequence $v^K$ converges weakly in $L^2((0,1), W^{m,2}(D,\mathbb{R}^d) \cap W^{1,2}_0(D,\mathbb{R}^d))$ to some $v \in L^2((0,1), W^{m,2}(D,\mathbb{R}^d) \cap W^{1,2}_0(D,\mathbb{R}^d))$ for $K \to \infty$. Then, by a standard weak lower semi-continuity argument we obtain
\[
\liminf_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \int_D \frac{\lambda}{2} (\text{tr}[v^K_k])^2 + \mu \text{tr}(\varepsilon[v^K_k])^2 + \gamma |D^m v^K_k|^2 \, dx 
= \liminf_{K \to \infty} \int_0^1 \int_D \frac{\lambda}{2} (\text{tr}[v])^2 + \mu \text{tr}(\varepsilon[v])^2 + \gamma |D^m v|^2 \, dx \, dt 
\geq \int_0^1 \int_D \frac{\lambda}{2} (\text{tr}[v])^2 + \mu \text{tr}(\varepsilon[v])^2 + \gamma |D^m v|^2 \, dx \, dt.
\]

It remains to verify that we can pass to the limit in (4.3) for $K \to \infty$ with $v$ also being the weak limit of $\tilde{v}^K$ in $L^2((0,1) \times D)$. This will indeed imply that $z$ is the weak material derivative for the image path $u$ and the velocity field $v$ fulfilling (1.4) and hence $(v,z) \in T_u L^2(D)$. To this end, the main difficulty is to prove the weak continuity of $(u,v) \mapsto u\text{div}(v)$. In [34, Theorem 2] (with the essential ingredient, which we actually required here, given in [34, Lemma 6]) it is shown that for the family of diffeomorphisms $\psi : [0,1] \to C^1(D)$ resulting from the transport
\[
\dot{\psi}(t,\cdot) = v(t, \psi(t,\cdot)) \tag{4.7}
\]
for some velocity field $v \in L^2((0,1), W^{m,2}(D) \cap W^{1,2}_0(D))$ and for given initial data $\psi(0) = 1$ the integral formula
\[
u(t,x) = u(0,\psi_{t,0}(x)) + \int_0^t z(s,\psi_{t,s}(x)) \, ds \tag{4.8}
\]
for an image path $u$, a function $z \in L^2((0,1), L^2(D))$ and for a.e. $x \in D$ with $\psi_{t,s} = \psi(s, (\psi(t,\cdot))^{-1})$ is equivalent to (1.4). We refer to [13, Lemma 2.2] for the existence of a unique solution $\psi$ of (4.7). From (4.2) we deduce that $(u^K, \tilde{v}^K, z^K)$ obeys
\[
u^K(t,x) = u^K(0,\psi^K_{t,0}(x)) + \int_0^t z^K(s,\psi^K_{t,s}(x)) \, ds, \tag{4.9}
\]
where $\psi^K_{t,s} = \psi^K(s,(\psi^K)^{-1}(t,\cdot))$ with $\psi^K : [0,1] \to C^1(D)$ denoting the time discrete family of diffeomorphisms induced by the motion field $\dot{\psi}^K$ and solving

$$
\dot{\psi}^K(t,x) = \tilde{\psi}^K(t,\psi^K(t,x))
$$

(4.10)

for all $x \in D$. In what follows, we will show strong convergence of $\psi^K$ to $\psi$, for which (4.7) holds. At first, we observe that $\|y^K_k(t,\cdot)\|_{C^{1,\alpha}(D)} \leq C(1 + K^{-1})\|v^K(t,\cdot)\|_{C^{1,\alpha}(D)}$ for $y^K_k(t,x) = x + (t-t_k)v^K_k(x)$ and $t \in [t_{k-1},t_k)$. By Sard’s theorem in Hölder spaces [7] and (4.6) we deduce that $\|x^K_k(t,\cdot)\|_{C^{1,\alpha}(D)} \leq C(1 + K^{-1})\|v^K(t,\cdot)\|_{C^{1,\alpha}(D)}$ for the inverse $x^K_k(t,\cdot) = y^K_k(t,\cdot)^{-1}$. Using the definition of $\tilde{v}^K$, the $C^{1,\alpha}$-estimate for the concatenation of $C^{1,\alpha}$-functions, and (4.6) we get

$$
\|\tilde{v}^K_k(t,\cdot)\|_{C^{1,\alpha}(D)} \leq C \|v^K(t,\cdot)\|_{C^{1,\alpha}(D)} \left(1 + K^{-1}\|v^K(t,\cdot)\|_{C^{1,\alpha}(D)}\right)
$$

The uniform boundedness of $v^K$ in $L^2((0,1),W^{m,2}(D))$ and the continuity of the embedding of $W^{m,2}(D)$ into $C^{1,\alpha}(D)$ imply that $\psi^K$ is uniformly bounded in $L^2((0,1),C^{1,\alpha}(D))$. Following [34, Lemma 7] (in a straightforward generalization for velocities uniformly bounded in $L^1((0,1),C^{1,\alpha}(D))$) one shows via Gronwall’s inequality that $\psi^K$ defined in (4.10) is uniformly bounded in $L^\infty((0,1),C^{1,\alpha}(D))$. Finally, using this bound and once again the $C^{1,\alpha}$-estimate for the concatenation of $C^{1,\alpha}$-functions we obtain from (4.10) the estimate

$$
\|\psi^K(t,\cdot) - \psi^K(s,\cdot)\|_{C^{1,\alpha}(D)} \leq \int_s^t \|v^K(r,\cdot)\|_{C^{1,\alpha}(D)} \, dr
$$

$$
\leq (t-s)^{\frac{1}{2}} \left(\int_s^t \|v^K(r,\cdot)\|_{C^{1,\alpha}(D)}^2 \, dr\right)^{\frac{1}{2}} \leq C(t-s)^{\frac{1}{2}},
$$

which proves that $\psi^K$ is uniformly bounded in $C^{0,\frac{1}{2}}((0,1),C^{1,\alpha}(D))$. Thus, for some $\beta$ with $0 < \beta < \min\{\frac{1}{2},\alpha\}$ and up to the selection of a subsequence $\psi^K$ converges strongly in $C^{0,\beta}((0,1),C^{1,\beta}(D))$ to some $\psi \in C^{0,\frac{1}{2}}((0,1),C^{1,\alpha}(D))$ and $\psi$ solves (4.7) (cf. [34, Theorem 9]). In addition, the mapping $\left(t \mapsto (\psi^K(t,\cdot))^{-1}\right)_{K \in \mathbb{N}}$, which solves (4.10) backward in time, is uniformly bounded in $C^{0,\frac{1}{2}}((0,1),C^{1,\alpha}(D))$ (cf. [34, Lemma 9]). Next, we obtain from (4.9) for functions $u^K$ with bounded energy $\mathcal{E}_K$ the following estimate:

$$
\|u^K(t+\tau,\psi^K(t+\tau,\cdot)) - u^K(t,\psi^K(t,\cdot))\|_{L^2(D)}^2
$$

$$
\leq \int_D \left(\int_t^{t+\tau} z^K(s,\psi^K(s,x)) \, ds\right)^2 \, dx
$$

$$
\leq \tau \|\det D((\psi^K)^{-1})\|_{L^\infty((0,1) \times D)} \int_t^{t+\tau} \|z^K(s,\cdot)\|_{L^2(D)}^2 \, ds
$$

$$
\leq C \tau \|z^K\|_{L^2((0,1) \times D)}^2 \leq C \tau
$$

(4.11)

for all $t \geq 0$, $\tau > 0$ with $t + \tau \leq 1$. The analogous estimate holds for $u^K, \psi^K$, and $z^K$ replaced by $u, \psi$, and $z$, respectively (cf. [34]). From this and the uniform smoothness of $\psi^K, \psi$ we deduce that for a subsequence (again denoted by $(u^K)_{K \in \mathbb{N}}$) $u^K(t) \to u(t)$ weakly in $L^2(D)$ for all $t \in [0,1]$. Indeed, this can be seen as follows. For an arbitrary test function
\[ \eta \in C^\infty(D), \ t \in (0, 1) \] and \( \tau > 0 \) sufficiently small (in the what follows for \( t = 0: t - \tau \) is replaced by \( t \) and for \( t = 1: t + \tau \) is replaced by \( t \)) we obtain

\[
\int_D (u^K(t,x) - u(t,x)) \eta(x) \, dx \\
= \int_{t-\tau}^{t+\tau} \int_D (u^K(t,x) - u^K(s,x)) \eta(x) - (u(t,x) - u(s,x)) \eta(x) \, dx \, ds \\
+ \int_{t-\tau}^{t+\tau} \int_D (u^K(s,x) - u(s,x)) \eta(x) \, dx \, ds .
\] (4.12)

Due to the weak convergence of \( u^K \to u \) in \( L^2((0,1) \times D) \) the second integral on the right-hand side of (4.12) vanishes as \( K \to \infty \). Setting

\[
\tilde{\eta}^K(t,y) = \eta(\psi^K(t,y)) \det D\psi^K(t,y) , \quad \tilde{\eta}(t,y) = \eta(\psi(t,y)) \det D\psi(t,y)
\]

we can rewrite the first term in the first integral on the right-hand side of (4.12) and get

\[
\int_{t-\tau}^{t+\tau} \int_D u^K(t,\psi^K(t,y)) \tilde{\eta}^K(t,y) - u^K(s,\psi^K(s,y)) \tilde{\eta}^K(s,y) \, dy \, ds \\
= \int_{t-\tau}^{t+\tau} \int_D (u^K(t,\psi^K(t,y)) - u^K(s,\psi^K(s,y))) \tilde{\eta}^K(t,y) \, dy \, ds \\
+ \int_{t-\tau}^{t+\tau} \int_D u^K(s,\psi^K(s,y)) (\tilde{\eta}^K(t,y) - \tilde{\eta}^K(s,y)) \, dy \, ds .
\] (4.13)

The second integral on the right-hand side of (4.13) vanishes due to the smoothness of \( \eta \) and \( \psi^K \) as \( \tau \to 0 \). Furthermore, using (4.11) the first integral can be estimated by

\[
\left\| \int_{t-\tau}^{t+\tau} \int_D (u^K(t,\psi^K(t,y)) - u^K(s,\psi^K(s,y))) \tilde{\eta}^K(t,y) \, dy \, ds \right\| \\
\leq \sup_{s \in [t-\tau,t+\tau]} \left\| u^K(t,\psi^K(t,\cdot)) - u^K(s,\psi^K(s,\cdot)) \right\|_{L^2(D)} \left\| \tilde{\eta}^K(t,\cdot) \right\|_{L^2(D)} \\
\leq C \tau^{\frac{1}{2}} \left\| \tilde{\eta}^K(t,\cdot) \right\|_{L^2(D)},
\]

and thus also vanishes for \( \tau \to 0 \). Analogous estimates apply to the remaining expression in (4.12) replacing \( u^K, \tilde{\eta}^K, \) and \( \psi^K \) by \( u, \tilde{\eta}, \) and \( \psi, \) respectively. Altogether, this proves \( u^K(t) \to u(t) \) weakly in \( L^2(D) \) for all \( t \in [0,1] \). Multiplying (4.9) with a test function \( \eta \in C^\infty(D) \) and integrating over \( D \) yields

\[
0 = \int_D u^K(t,x) \eta(x) \, dx - \int_D u^K(0,\psi^K_{t;0}(x)) \eta(x) \, dx - \int_0^t \int_D z^K(s,\psi^K_{t,s}(x)) \eta(x) \, dx \, ds \\
= \int_D u^K(t,x) \eta(x) \, dx - \int_D u^K(0,y) \eta(\psi^K_{t;0})^{-1}(y) (\det D\psi^K_{t;0})^{-1}(\psi^K_{t;0})^{-1}(y) \, dy \\
- \int_0^t \int_D z^K(s,y) \eta(\psi^K_{t;0})^{-1}(y) (\det D\psi^K_{t;0})^{-1}(\psi^K_{t;0})^{-1}(y) \, dy .
\] (4.14)

Based on the weak convergence of \( u^K, z^K \) and the strong convergence of \( t \mapsto (\psi^K(t,\cdot))^{-1} \)
and \( \psi^K \) we can pass to the limit in (4.14) and obtain
\[
0 = \int_D u(t,x) \eta(x) \, dx - \int_D u(0,y) \eta((\psi_{t,0})^{-1}(y)) \, dy \\
- \int_0^t \int_D z(s,y) \eta((\psi_{t,s})^{-1}(y)) \, dy \\
= \int_D u(t,x) \eta(x) \, dx - \int_D u(0,(\psi_{t,0}(x))) \, dx - \int_0^t \int_D z(s,(\psi_{t,s}(x))) \eta(x) \, dx \, ds,
\]
which shows that \( u \) and \( z \) fulfill (4.8) for a.e. \( x \in D \). Since (4.8) is equivalent to (1.4), this finally proves (1.4).

[limsup—estimate] Consider an image curve \( u \in L^2((0,1) \times D) \) with \( u(t) \in \mathcal{I} \) for all \( t \in [0,1] \). Without any restriction we assume that the energy
\[
\mathcal{E}[u] = \int_0^1 \int_D L[v,v] + \frac{1}{2} |v|^2 \, dx \, dt
\]
is bounded, where \( v \in L^2((0,1), W^{m,2}(D) \cap W^{1,2}_0(D)) \) and \( z \in L^2((0,1) \times D) \) are an optimal velocity field and a corresponding weak material derivative, respectively. Now, we define an approximate, piecewise constant (in time) velocity field
\[
v^K_k = v^K := K \int_{t_{k-1}}^{t_k} v \, dt
\]
for \( k = 1, \ldots, K \) and again denoting \( t_k = \frac{K}{N} \).

Obviously, \( v^K \) converges to \( v \) in \( L^2((0,1), W^{m,2}(D)) \). We denote by \( \psi^K \) the associated flow of diffeomorphism generated by the flow equation \( \psi^K(t,x) = v^K(t,\psi^K(t,x)) \) with \( \psi^K(0,x) = x \) and by \( \psi^K = \psi^K(s,(\psi^K)^{-1}(t,\cdot)) \) the induced relative deformation from time \( t \) to time \( s \). From this, we also obtain the underlying vector of consecutive deformations \( \Phi^K = (\phi^K_1, \ldots, \phi^K_K) \) with \( \phi^K_k = \psi^K_{t_k,t_{k-1}}. \) Following [13] we easily verify that the evolution equation for \( \psi^K \), the uniform smoothness of \( \psi^K \) and the bound on the energy \( \mathcal{E}[u] \) imply that \( \psi^K \) is uniformly bounded in \( C^{0,\frac{1}{2}}([0,1], C^{1,\alpha}(\bar{D})) \) (cf. the proof of the lim inf-estimate above).

Next, the approximate discrete image path \( u^K = (u^K_0, \ldots, u^K_K) \) is defined by a discrete counterpart of (4.8), namely
\[
u^K_k(x) = u(0, \psi^K_{t_k,0}(x)) + \int_0^{t_k} z(s, \psi^K_{t_k,s}(x)) \, ds
\]
for \( k = 0, \ldots, K \). The boundedness of the energy implies that \( u^K(t_k,\cdot) \) for \( k = 0, \ldots, K \) are well-defined and uniformly bounded intensity functions in \( L^2(D) \). Indeed, we also have that \( u^K(t_k,\cdot) \in \mathcal{I} \). This can be seen as follows. For a minimizer of the continuous path energy on \( L^2((0,1) \times D) \) with \( u(0) \in \mathcal{I} \) and \( u(1) \in \mathcal{I} \), Trouvé and Younes give in [34, Theorem 4] and [34, Theorem 2] a direct representation of the intensity function, namely
\[
u(t,\cdot) = u(0, \psi^{-1}(\cdot)) + \left( z_0 \int_0^1 (\det D\psi(s))^{-1} \, ds \right) \circ \psi^{-1}
\]
for some \( z_0 \in L^2(D) \) and \( \psi(t) = \psi(t,\cdot) \) the underlying flow of diffeomorphisms. Now, evaluating this equation for \( t = 1 \) gives
\[
z_0 = (u(1, \psi(1,\cdot)) - u(0)) \left( \int_0^1 (\det D\psi)^{-1}(s) \, ds \right)^{-1}.
\]
Hence, \( z_0 \in \mathcal{I} \) and thus the claim that \( u^K \in \mathcal{I} \) is indeed a consequence of (4.15).

Next, taking into account that \( u^K \in \mathcal{I} \) for \( k = 0, \ldots, K \) and using (4.1) one obtains \( u^K = U_K[u^K, \Phi^K] \) as the requested approximation of \( u \) for given \( K \in \mathbb{N} \). At first, we verify that \( \limsup_{K \to \infty} E_K[u^K] \leq E[u] \). From the minimizing property of \( U_K[u^K, \Phi^K] \) we deduce

\[
E_K[u^K] = E_K[u^K] \leq K \sum_{k=1}^{K} \int_D W(D\phi_k^K) + \gamma|D^m\psi_k^K|^2 + \frac{1}{\delta}|u_k^K - \phi_k^K - u_{k-1}^K|^2 \, dx.
\]

Using the Cauchy-Schwarz inequality we derive from (4.15)

\[
\int_D |u_k^K \circ \phi_k^K(x) - u_{k-1}^K(x)|^2 \, dx = \int_D \int_{t_{k-1}}^{t_k} |z(s, \psi_{k-1,s}(x))| \, ds \, dx \\
\leq \frac{1}{K} \int_{t_{k-1}}^{t_k} \int_D |z(s, x)|^2 \, dx \, ds \leq \frac{1}{K} \int_{t_{k-1}}^{t_k} \int_D |z(s, x)|^2 \, dx \, ds + \frac{C}{K^{3/2}},
\]

where we have taken into account the estimate \(|1 - \det D(\psi_{k-1,s})^{-1}(x)| \leq C K^{-1/2} \), which follows from the uniform bound for \( \psi^K \) in \( C^{0,1/2}([0,1], C^{1,1}((\bar{D})) \). Furthermore, we obtain via Taylor expansion and the consistency assumption (W3)

\[
\int_D W(D\psi_{k-1,t_k}) + \gamma|D^m\psi_{k-1,t_k}|^2 \, dx \\
\leq \int_D \frac{1}{2K^2} D^2 W(1)(Dv_k^K, Dv_k^K) + \frac{\gamma}{K^2}|D^m\psi_k^K|^2 \, dx + C \int_D \frac{1}{K^3} |Dv_k^K|^3 \, dx \\\n= \frac{1}{K^2} \int_D L[v_k^K, v_k^K] \, dx + \frac{C}{K^3} \int_D |Dv_k^K|^3 \, dx.
\]

The definition of \( \psi_k^K \) together with Jensen’s inequality implies

\[
\int_D L[v_k^K, v_k^K] \, dx \leq K \int_D \int_{t_{k-1}}^{t_k} L[v, v] \, dt \, dx.
\]

To estimate the remainder of the Taylor expansion we proceed as follows. At first, we obtain

\[
\|v_k^K\|_{C^1(\bar{D})} \leq C \sum_{l=1}^{K} \|v_l^K\|_{W^{m,2}(D)} \leq C K \int_0^1 \|v(t, \cdot)\|_{W^{m,2}(D)} \, dt \leq C K
\]

using the Sobolev embedding theorem together with the Cauchy-Schwarz inequality and the boundedness of the energy \( E[u] \). Hence, \( \max_{k=1,\ldots,K} \|v_k^K\|_{C^1(\bar{D})} \leq C K^{1/2} \), which implies

\[
\frac{1}{K} \sum_{k=1}^{K} \int_D |Dv_k^K|^3 \, dx \leq \frac{1}{K} \sum_{k=1}^{K} \|v_k^K\|_{C^1(\bar{D})} \int_D \left( K \int_{t_{k-1}}^{t_k} Dv(t, x) \, dt \right)^2 \, dx \\
\leq C K^{1/2} \frac{K^2}{K} \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} |Dv(t, x)|^2 \, dt \, dx \leq C K^{3/2}.
\]

From these estimates we finally deduce

\[
E_K[u^K] \leq \int_0^1 \int_D L[v, v] + \frac{1}{\delta} |z|^2 \, dx \, dt + CK^{-1/2} + \frac{C}{\delta} K^{-1/2}.
\]
We are still left to demonstrate that $u^K \to u$ in $L^2((0,1) \times D)$. To see this, we first observe that by the theorem of Arzelà-Ascoli and after selection of a subsequence $\psi^K$ converges to $\psi$ in $C^{0,\beta}([0,1], C^{1,\alpha} (\bar{D}))$ with $\beta < \frac{1}{2}$ and $\alpha < m - \frac{d}{2} - 1$. From this and the quantitative control of the inverse of the diffeomorphisms (cf. [34, Lemma 9]) we deduce that $\psi^K$, its inverse, and also $D\psi^K_{t,s}$ converge uniformly in $x, t$, and $s$. Thus, we get that for every $t \in (0,1)$$$
abla z(\cdot, \psi^K_{t,s} \cdot) - \nabla z(\cdot, \psi_{t,s} \cdot))\|_{L^2((0,1) \times D)} \to 0, \quad \|u(0, \psi^K_{t,s}(-)) - u(0, \psi_{t,s}(-))\|_{L^2(D)} \to 0$$for $K \to \infty$. Indeed, in case of the first claim we argue as follows. Due to the uniform bound on $z$ in $L^2((0,1) \times D)$ we only have to show that $\int_0^1 \int_D z(s, \psi^K_{t,s}^{-1}(x)) \eta(s, x) \, dx \, ds \to \int_0^1 \int_D z(s, \psi_{t,s}^{-1}(x)) \eta(s, x) \, dx \, ds$ for all $\eta \in C^\infty_c((0,1) \times D)$ and $q = 1,2$. This is easily seen via integral transform, i.e.$$
abla z(\cdot, \psi^K_{t,s} \cdot) - \nabla z(\cdot, \psi_{t,s} \cdot))\|_{L^2((0,1) \times D)} \to 0, \quad \|u(0, \psi^K_{t,s}(-)) - u(0, \psi_{t,s}(-))\|_{L^2(D)} \to 0$$for $K \to \infty$. The argument for $u(0, \cdot)$ is analogous. Hence, we can pass to the limit on the right-hand side of (4.15) and achieve in analogy to corresponding argument in the proof of the lim inf-estimate$$u^K = \left( (t, x) \mapsto u(0, \psi^K_{t,0}(x)) + \int_0^t z(s, \psi^K_{t,s}(x)) \, ds \right)$$
abla z(\cdot, \psi^K_{t,s} \cdot) - \nabla z(\cdot, \psi_{t,s} \cdot))\|_{L^2((0,1) \times D)} \to 0, \quad \|u(0, \psi^K_{t,s}(-)) - u(0, \psi_{t,s}(-))\|_{L^2(D)} \to 0$$for $K \to \infty$. The argument for $u(0, \cdot)$ is analogous. Hence, we can pass to the limit on the right-hand side of (4.15) and achieve in analogy to corresponding argument in the proof of the lim inf-estimate$$u^K = \left( (t, x) \mapsto u(0, \psi^K_{t,0}(x)) + \int_0^t z(s, \psi^K_{t,s}(x)) \, ds \right)$$where the convergence is in $L^2((0,1) \times D)$. 

**Theorem 4.2 (Convergence of discrete geodesic paths).** Let $u_A, u_B \in I$ and suppose that (W1-3) holds. Furthermore, for every $K \in \mathbb{N}$ let $u^K$ be a minimizer of $E_K$ subject to $u^K(0) = u_A, u^K(1) = u_B$. Then, a subsequence of $(u^K)_{K \in \mathbb{N}}$ converges weakly in $L^2((0,1) \times D)$ to a minimizer of the continuous path energy $E$ and the associated sequence of discrete energies converges to the minimal continuous path energy.

**Proof.** At first, as in the proof of Theorem 4.1, we make use of the direct representation formula $u(t, \cdot) = u_A \circ \psi(t)^{-1} + \left( z_0 \int_0^t (\det D\psi(s))^{-1} \, ds \right) \circ \psi(t)^{-1}$ where $z_0 = (u_B \circ \psi(1) - u_A) \left( \int_0^1 (\det D\psi)^{-1}(s) \, ds \right)^{-1}$ and thus$$u(t, \psi(t)) = u_A + (u_B \circ \psi(1) - u_A) \left( \int_0^1 (\det D\psi)^{-1}(s) \, ds \right)^{-1}$$for all $t \in [0,1]$. This immediately implies $u(t, \cdot) \in I$ for all $t \in [0,1]$. The remainder of the proof is standard in $\Gamma$-convergence theory (cf. [9]). Let $u^K \in \text{argmin} E_K$, where we minimize over all admissible image paths subject to the boundary conditions in time. From Remark 3.3 (ii) we deduce that $u^K$ is uniformly bounded in $L^2((0,1) \times D)$. Hence, there exists a subsequence, again denoted by $(u^K)_{K \in \mathbb{N}}$, with $u^K \rightharpoonup u$ (weakly) in $L^2((0,1) \times D)$
to some \( u \in L^2((0, 1) \times D) \). Now, let us assume that there is an image path \( \tilde{u} \) with \( \mathcal{E}[\tilde{u}] < \mathcal{E}[u] \). Then, by the \( \lim \sup \)-estimate of Theorem 4.1 there exists a sequence \( (\tilde{u}^K)_{K \in \mathbb{N}} \) with \( \tilde{u}^K \in L^2((0, 1) \times D) \) such that \( \lim \sup_{K \to \infty} \mathcal{E}_K[\tilde{u}^K] \leq \mathcal{E}[\tilde{u}] \) and together with the \( \lim \inf \)-estimate we obtain

\[
\mathcal{E}[u] \leq \lim \inf_{K \to \infty} \mathcal{E}_K[u^K] \leq \lim \sup_{K \to \infty} \mathcal{E}_K[\tilde{u}^K] \leq \mathcal{E}[\tilde{u}]
\]

which is a contradiction. Hence, \( u \) minimizes the continuous path energy over all admissible image paths. \( \square \)

5. Spatial discretization. We consider a regular quadrilateral grid on a two-dimensional, rectangular image domain \( D \) consisting of cells \( C_m \) with \( m \in I_C \), where \( I_C \) is the index set of all cells. Based on this grid, we define the finite element space \( V_h \) of piecewise bilinear continuous functions (cf. [8]) and denote by \( \{\Theta^i\}_{i \in I_N} \) the set of basis functions, where \( I_N \) is the index set of all grid nodes \( x_i \). Now, we investigate spatially discrete deformations \( \Phi_k : D \to \mathbb{R} \) with \( k = 1, \ldots, K \) and spatially discrete image maps \( U_k : D \to \mathbb{R} \) with \( U_k \in V_h \) and \( U_0 = U_A = I_h u_A \). \( U_K = U_B = I_h u_B \).

Here, \( I_h \) denotes the nodal interpolation operator. Given a finite element function \( W \in V_h \), we denote by \( W = (W(x_i))_{i \in I_N} \) the corresponding vector of nodal values. Furthermore, we define a fully discrete counterpart \( E_{K,h} \) of the so far solely time discrete path energy \( E_{K} \) as follows

\[
E_{K,h}([U_0, \ldots, U_K]) := \min_{\Phi_k \in V_h^K, \Phi_k|_{\partial D} = \Gamma} \sum_{k=1}^K E_{K,h}([U_0, \ldots, U_K], (\Phi_1, \ldots, \Phi_K)).
\]

Here, \( E_{K,h}^{D}([U_0, \ldots, U_K], (\Phi_1, \ldots, \Phi_K)) \) is the discrete counterpart of \( E_{K}^{D} \) and obtained by approximating the integrals of \( E_{K}^{D} \) on each cell with the Simpson quadrature rule.

Next, we study the numerical minimization of the fully discrete energy \( E_{K,h}^{D} \) for fixed \( (\Phi_1, \ldots, \Phi_K) \). For \( m \in I_C \) the Simpson quadrature takes into account nine quadrature points and let \( x_q^m \) denote the \( q \)-th quadrature point in \( C_m \), and \( w_q^m \) the corresponding quadrature weight for \( q \in \{0, \ldots, 8\} \). Then, the entries of a weighted mass matrix \( M_h[\Phi, \Psi] = (M_h[\Phi, \Psi]_{i,j})_{i,j \in I_N} \) with basis functions being transformed via deformations \( \Phi, \Psi \) and evaluated via a tensor product Simpson quadrature are given by

\[
M_h[\Phi, \Psi]_{i,j} := \sum_{l \in I_C} \sum_{q=0}^8 w_q^l \Theta^l(\Theta^l \circ \Phi)(x_q^l) (\Theta^l \circ \Psi)(x_q^l).
\]

To evaluate the entries of this matrix numerically, we use cell-wise assembly. For \( m \in I_C \), let \( \Theta_m^\alpha \) denote the basis function in the cell \( C_m \) with local index \( \alpha \in \{0, 1, 2, 3\} \) and \( I(m, \alpha) \) the global index corresponding to the local index \( \alpha \) in the cell \( C_m \), i.e. \( \Theta_{I(m, \alpha)} = \Theta_m^\alpha \) on \( C_m \).

The cell-wise assemble procedure works as follows. First, \( M_h[\Phi, \Psi] \) is initialized as the zero matrix. Then, for the every \( l \in I_C \) and every \( q \in \{0, \ldots, 8\} \) one identifies the cells \( C_m, C_m' \) with \( \Phi(x_q^l) \in C_m \) and \( \Psi(x_q^l) \in C_m' \), respectively. Finally, for all pairs of local indices \( (\beta, \beta') \) with \( \beta, \beta' \in \{0, 1, 2, 3\} \) one adds \( w_q^l \Theta_m^l(\Phi(x_q^l)) (\Psi(x_q^l)) \) to \( M_h[\Phi, \Psi]_{I(m, \beta), I(m', \beta')} \).

Now, we are in the position to derive a linear system of equations for the vector \( \tilde{U} = (\tilde{U}_1, \ldots, \tilde{U}_{K-1}) \) of images that describes a minimizer of \( E_{K,h}^{D} \) for a fixed vector of spatially discrete deformations \( \Phi = (\Phi_1, \ldots, \Phi_K) \). Indeed, we can rewrite the last term in the energy
\[ E^D_{K,h} \] as follows

\[
\sum_{k=1}^{K} \sum_{l \in I_C} \sum_{q=0}^{8} w_q^l (|U_k \circ \Phi_k - U_{k-1}|^2) (x_q^l)
\]

\[
= \sum_{k=1}^{K} (M_h[\Phi_k, \Phi_k] \bar{U}_k - \bar{U}_k - 2M_h[\Phi_k, 1] \bar{U}_k \circ \bar{U}_{k-1} + M_h[1, 1] \bar{U}_{k-1} \cdot \bar{U}_{k-1} ).
\]

From this, we obtain for the variation of the energy \( E^D_{K,h} \) with respect to the \( k \)-th image map

\[
\partial U_k E^D_{K,h} = 2 (M_h[\Phi_k, \Phi_k] + M_h[1, 1]) \bar{U}_k - 2M_h[\Phi_k, 1] \bar{U}_{k-1} - 2M_h[\Phi_{k+1}, 1] \bar{U}_{k+1}
\]

for \( k = 1, \ldots, K - 1 \). For a fixed set of deformations a necessary condition for \( \bar{U} \) to be a minimizer of \( E^D_{K,h} \) is that \( \bar{U} \) solves the block tridiagonal system of linear equations \( A[\Phi] \bar{U} = R[\Phi] \), where \( A[\Phi] \) is formed by \( (K - 1) \times (K - 1) \) matrix blocks \( A_{k,k'} \in \mathbb{R}^{I_N \times I_N} \) and \( R[\Phi] \) consists of \( K - 1 \) vector blocks \( R_k \in \mathbb{R}^{I_N} \) with

\[
A_{k,k-1} = -M_h[\Phi_k, 1]^T, \quad A_{k,k} = M_h[\Phi_k, \Phi_k] + M_h[1, 1], \quad A_{k,k+1} = -M_h[\Phi_{k+1}, 1],
\]

\[
R_1 = M_h[\Phi_1, 1]^T \bar{U}_A, \quad R_2 = R_3 = \ldots = R_{K-2} = 0, \quad R_{K-1} = M_h[\Phi_K, 1] \bar{U}_B.
\]

The energy \( \sum_{l \in I_C} \sum_{q=0}^{8} w_q^l (|U_k \circ \Phi_k - U_{k-1}|^2) (x_q^l) \) is convex in \( U_k \) and strictly convex in \( U_{k-1} \). Hence, \( E^D_{K,h} \) is strictly convex in \( U \) and there is a unique minimizer \( U = U[\Phi] \) for fixed \( \Phi \). This implies that \( A \) is invertible and by solving the linear system \( A[\Phi] \bar{U} = R[\Phi] \) one computes this unique minimizer. Numerically, the corresponding system of linear equations (cf. line 12 of Algorithm 1) is solved with a conjugate gradient method with diagonal preconditioning.

For fixed \( U \), the deformations \( \Phi_1, \ldots, \Phi_K \) are independent of each other and thus can be updated separately. The actual minimization of \( E^D_{K,h} \) with respect to \( \Phi_k \) (the numerical solution of a simple registration problem) is implemented based on a step size controlled Fletcher-Reeves nonlinear conjugate gradient descent scheme with respect to a regularized \( H^1 \)-metric on the space of deformations [33]. Furthermore, we take into account a cascadic approach starting with a coarse time discretization and then successively refine the time discretization. In each step of this approach, we minimize the discrete path energy and perform a prolongation to the next finer level of the time discretization. The prolongation is based on the insertion of new midpoint images between every pair of consecutive images. To this end we compute an optimal deformation between a pair of images and insert the middle image of the resulting warp. To improve the robustness of the algorithm, we additionally use a Gaussian filter with variance \( \sigma^2 = 5 \cdot 10^{-3} \) to pre-filter the input images and damp noise. The resulting alternating minimization algorithm is summarized in Algorithm 1.

In the applications, it is frequently appropriate to ensure that deformations are not restricted too much by the Dirichlet boundary condition \( \Phi = 1 \) on \( \partial D \). This can practically be obtained by enlarging the computational domain and considering an extension of the image intensities with a constant gray or color value or by taking into account natural boundary conditions for the deformations. This can theoretically be justified by adding constraints on the mean deformation and the angular momentum. In our computations, such constraints are usually not required to avoid an unbounded rigid body motion component of the numerical solution.
K = 4 is set to 10 of the deformation energy to comply with (W2) and (W3). However, in case of the definition (6.2) the regularization term for the Γ consisting of a simple thin plate spline regularization and the most basic fidelity term (cf. [25]). In fact, this retrieves a very basic model for the registration of the two images slices of a 3D magnetic resonance tomography of a human brain.

The simplified model is associated with the quadratic form
\[ L[v(t), v(t)] := Dv : Dv + \gamma \Delta v \cdot \Delta v, \]
where \( \gamma > 0 \). A choice for the discrete energy, which is consistent with this quadratic form, is given by
\[ W[u, \tilde{u}] = \min_{\phi} \int_D D\phi : D\phi + \gamma \Delta \phi \cdot \Delta \phi + \frac{1}{\delta} |\tilde{u} \circ \phi - u|^2 \, dx. \]
In fact, this retrieves a very basic model for the registration of the two images \( u \) and \( \tilde{u} \) consisting of a simple thin plate spline regularization and the most basic fidelity term (cf. [25]).

Let us emphasize that in the spatially continuous setting both the existence theory and the \( \Gamma \)-convergence result require the full set of assumptions. In particular, the definitions (6.1) for \( L[\cdot, \cdot] \) and (6.2) for \( W \) (contrary to the full model with \( W \) proposed in (2.3)) do not comply with (W2) and (W3). However, in case of the definition (6.2) the regularization term of the deformation energy \( W \) is quadratic and enables a significant speedup of the algorithm compared to the theoretically justified fully nonlinear model. We compare both models in our first example and use the simplified model in all other applications. The parameter threshold is set to \( 10^{-6} \) in the algorithm.

Figure 6.1 depicts a discrete geodesic path obtained with the full model (with parameters \( K = 4, \delta = 10^{-2}, \lambda = 1, \mu = \frac{1}{2}, q = r = \frac{1}{2}, s = \frac{1}{2} \) and \( \gamma = 10^{-5} \)) and with the simplified model (with parameters \( K \in \{4, 16\}, \gamma = 10^{-3}, \delta = 10^{-1} \)), where \( u_A \) and \( u_B \) are different slices of a 3D magnetic resonance tomography of a human brain.

Figure 6.2 shows a geodesic path between two faces from female portrait paintings\(^1\)

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\(^1\)first painting by A. Kauffmann (public domain, see [http://commons.wikimedia.org/wiki/File:](http://commons.wikimedia.org/wiki/File:))
$K = 4$, using (2.1), (2.3) with $\delta = 10^{-2}$, $\lambda = 1, \mu = \frac{1}{2}, q = r = \frac{1}{2}, s = \frac{1}{2}, \gamma = 10^{-5}$.

$K = 4$, using (6.2) with $\gamma = 10^{-3}, \delta = 10^{-1}$.

$K = 16$, using (6.2) with $\gamma = 10^{-3}, \delta = 10^{-1}$.

Fig. 6.1. Metamorphosis for two slices of a MRT data set of a human brain (data courtesy of H. Urbach, Neuroradiology, University Hospital Bonn). We compare the original model (first row) with the simplified model and $K = 4$ (second row), $K = 16$ (third to fifth row).

computed with the simplified model with parameters $\gamma = 10^{-3}$ and $\delta = 10^{-2}$. The local contributions $\mathcal{E}^D_K[(U_{k-1}, U_k), \Phi_k]$ for $k = 1, \ldots, K$ of the total energy and its components are shown in Figure 6.3. Note that the method seems to prefer an approximate equidistribution of the total path energy in time.

Finally, we consider time-discrete geodesic path in the space of color images. To this end, we take into account a straightforward generalization of the model for scalar (gray) valued image maps to vector-valued image maps. One can even enhance the model with further channels. Such additional channels can represent segmented regions of the images, which one would like to ensure to be properly matched by transport and not by blending of intensities. The only required modification of the method is that $|u_{k+1} \circ \phi_{k+1} - u_k|$ is now the Euclidean norm of the (extended) color vector. As an application, we considered the metamorphosis between two self-portraits by van Gogh (see Figure 6.5)\(^2\). Since the background colors of both self-portraits differ considerably in the RGB color space, we adjusted the background color

\( K = 4, \) using (6.2) with \( \gamma = 10^{-3}, \delta = 10^{-2} \)

\( K = 16, \) using (6.2) with \( \gamma = 10^{-3}, \delta = 10^{-2} \)

---

**Fig. 6.2.** Metamorphosis between two faces from female portrait paintings.

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**Fig. 6.3.** Energy contributions of the regularization functional \( \int_D \Phi_k \cdot D \Phi_k + \gamma \Delta \Phi_k \cdot \Delta \Phi_k \, dx \) (red) and the matching functional \( \frac{1}{2} \int_D \| \tilde{U}_k \circ \Phi_k - U_{k-1} \|_2^2 \, dx \) (green) for the discrete geodesic path in Figure 6.2 with \( K = 4 \) (left) and \( K = 16 \) (right).

---

The time-discrete geodesic path for the van Gogh self-portraits is shown in Figure 6.6 for \( K = 8 \) along with the temporal change of the fourth channel. Again, we used the simplified model with parameters \( \gamma = 10^{-3} \) and \( \delta = 10^{-2} \). Figure 6.4 depicts the pullback \( u_{RGB}^B \circ \Phi \) along the flow induced deformation \( \Phi = \Phi_K \circ \Phi_{K-1} \circ \ldots \circ \Phi_1 \) corresponding to the geodesics in Figure 6.6. Finally, Figure 6.7 visualizes the deformations and the corresponding accumulated weak material derivative along the discrete geodesic path. The color wheel on the lower left in the first row indicates both the direction and the magnitude of the discrete velocities \( K(\Phi_k - 1) \). Obviously, the motion field is not constant in time. Furthermore, to visualize...
the change of the image intensity along motion paths, the accumulated weak material derivative \( Z_l (l = 1, \ldots, 8) \) with \( Z_l = K \sum_{k=1}^{l} (U_k \circ \Phi_k - U_{k-1}) \circ X_{k-1} \) using the notation (3.2) is plotted using an equal rescaling for all \( l \).

7. Conclusions and outlook. We have developed a robust and effective time discrete approximation for the metamorphosis approach to compute shortest paths in the space of images. Thereby, the underlying discrete path energy is a sum of classical image matching functionals. The approach allows for edge type singularities in the input images. We have proven existence of minimizers of the discrete path energy and convergence of minimizing discrete paths to a continuous path, which minimizes the continuous path energy. This analysis is based on a combination of the variational perspective of (discrete) geodesics as minimizers of the continuous (1.5) and discrete path energy (2.2), respectively, with the continuous ((4.7), (4.8)) and discrete flow perspective ((4.10), (4.9)). In particular, this combination is the basis of a compensated compactness argument for the weak material derivative. Using a finite element ansatz for the spatial discretization, a numerical algorithm has been presented to compute discrete geodesic paths. Qualitative properties of the algorithm are discussed for three different examples including an application to multi-channel images. Particularly interesting future research directions are
- the use of duality techniques in PDE constraint optimization to derive a Newton type scheme for the simultaneous optimization of the set of deformations and the set of images associated with the discrete path,
- a fully fledged discrete geodesic calculus based on the general procedure developed in [28, 29] and including a discrete logarithmic map, a discrete exponential map, and a discrete parallel transport, and
- a concept for discrete geodesic regression and geometric, statistical analysis in the space of images.

Furthermore, the close connection to optimal transportation offers interesting perspectives, which should be exploited.

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REFERENCES

FIG. 6.6. Metamorphosis between two “van Gogh self-portraits” using the energy (6.2) for $K = 8$ and $\delta = 10^{-2}$ including the fourth (segmentation) channel (bottom row).

Discrete motion fields $K(\Phi_k - 1)$ (first row) and accumulated weak material derivative $Z_l$ (second row) for $k = 1, \ldots, 9$.


